Partial differential equations on random surfaces

by

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Declarations

The first chapter of this thesis will based on upon the paper [16] that I wrote, with discussion and collaboration with Ana Djurdevac, F.U. Berlin. I declare that to the best of my knowledge, that the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university.

Abstract

In this thesis, we will begin by analysing the domain mapping method for elliptic partial differential equations defined over random surfaces and random bulk-surface systems. In particular, we will begin by deriving expressions for the pull-back of geometric quantities and tangential differential operators defined over a random surface, onto a deterministic reference surface via a prescribed domain mapping. These calculations will allow for the original considered elliptic equations posed either over a random surface or a random bulk-surface system, to be reformulated respectively onto a deterministic reference surface and a deterministic bulk-surface system, and lead to the consideration of stochastic elliptic equations posed over a deterministic domain. An abstract analysis will subsequently be presented to treat the arising equations, and a numerical scheme based upon a piecewise linear finite element discretisation and a linear approximation of the curved reference domain will be presented and analysed in the abstract setting. Optimal error estimates will be derived and the convergence rates will be numerical verified in the case of a model elliptic surface equation and a coupled bulk-surface system. In the following chapter, we extend the application of the domain mapping method to the consideration of advection-diffusion equations posed over randomly evolving surfaces and randomly evolving bulk-surface systems. This will similarly entail first deriving expressions for the pull-back of time-dependent quantities, such as the material derivative, onto the reference domain, which will allow for a reformulation of the considered partial differential equations posed over the random domain, onto the reference domain to take place. After which, an abstract analysis of the stochastic partial differential equations which arise after reformulating the original advection-diffusion equations onto the deterministic reference domain will be presented. A numerical scheme based upon a piecewise linear finite element approximation coupled with a single level Monte-Carlo sampling, will subsequently be presented and analysed in the abstract setting and optimal error estimates will be derived. The convergence rate are subsequently numerically verified. The thesis will then conclude with a future outlook on the applications of the domain mapping method, in particular examining how the domain mappping method may applied to a

Hele-Shaw problem and a two-phase Stefan problem both posed over a random surface.

Chapter 1

Introduction

One of the primary aims of this thesis is to extend and analyse applications of the domain mapping method to the consideration of partial differential equations posed over random curved domains, such as random surfaces and random bulk-surface systems. The key concept of the domain mapping method, which will be discussed in greater depth later, is to exploit a stochastic parametric representation of the random domain, by reformulating the initial partial differential equations posed over the random domain onto a fixed deterministic reference domain. The result of such a transformation, is the consideration of stochastic partial differential equations posed over a deterministic domain, for which there exists extensive literature on the subject of their corresponding analysis and numerical treatment. One of the key challenges and contributions of this thesis, is the neccesary computations for the pull-back of tangential differential operators, geometric quantities and tangential velocity fields defined over a random surface, onto a deterministic reference surface via a given stochastic domain mapping. These calculations will allow for the consideration of the domain mapping method to be applied to partial differential differential equations posed over random curved domains. The potential applications for the domain mapping method is further extended in Chapter 3, to the consideration of partial differential equations posed over randomly evolving curved domains. We will now provide a general outline for the thesis.

In Chapter 2, we will begin by introducing the domain mapping method for a model elliptic partial differential equation posed over a random flat domain, to illustrate the key concepts of the method first proposed in [101], to analyse random domain problems. After which, we will then proceed by deriving expressions for the pull-back of geometric quantities and tangential differential operators defined over a random surface, onto a deterministic reference surface via a prescribed domain mapping. These calculations will allow for the domain mapping method, originally proposed for flat domains with random boundaries [101], to be extended to applications in which the random domain is curved. We will then introduce two model elliptic problems, posed respectively over a random surface and a random coupled bulk-surface system, and derive and analyse the stochastic elliptic partial differential equations on the corresponding deterministic reference domains, obtained from the reformulation process. Regularity results including uniform H^2-estimates, will be established for the pathwise solutions of the reformulated equations for each of the considered problems. We will then continue by presenting an abstract analysis of the general form of the stochastic elliptic equations which will arise after reformulating an initial elliptic random domain problem onto a respective reference domain. A numerical scheme based upon a piecewise linear finite element discretisation with a linear approximation of the curved reference domain, will subsequently be presented and analysed in the abstract setting. Optimal order error estimates will be derived and numerically verified for a model elliptic equation on a random surface and additionally a coupled elliptic system on a random bulk-surface.

In Chapter 3, we further extend the potential applications of the domain mapping method, by considering advection-diffusion equations posed over randomly evolving surfaces and randomly evolving bulk-surface systems. We first begin the chapter, by providing a description of the geometry of the randomly evolving surface and after which derive an advection-diffusion equation on the randomly evolving surface via a conservation law. Following on from this, we outline a general framework in which the domain mapping method may be applied to partial differential equations defined over randomly evolving curved domains, in which we will treat the reference domain selected in its full generality, by allowing for its deterministic evolution in time. Following the general framework presented, we will continue by deriving expressions for the pull-back of time-dependent quantities, such as the material derivative and tangential velocity fields, onto the reference domain via the prescribed domain mapping. These calculations will be necessary in order to reformulate the considered advection-diffusion equations onto their respective curved reference domains. We will then proceed by describing two model problems consisting of an advection-diffusion equation on a randomly evolving surface, and a coupled advection-diffusion system posed over a randomly evolving bulk-surface. With our previous computations, we reformulate the respective problems onto their corresponding deterministic reference domains and subsequently analyse the resulting stochastic partial differential equations. Existence and uniqueness of the pathwise solutions to each of the reformulated problems will be proved alongside regularity results, and uniform bounds on the solutions will be further derived based upon the assumptions imposed on the respective domain mappings. We will then continue by presenting an abstract analysis of the general form of the reformulated advection-diffusion equations which arise after the domain mapping method has been applied. In particular, we analyse in the abstract setting, a finite element discretisation coupled with a Monte-Carlo sampling, and establish optimal error bounds based upon previously derived results for the deterministic case. Two finite element discretisations will then be proposed for our two model problems, and will be shown to satisfy all the stated assumptions presented in the abstract framework required to derive an error estimate. The chapter will subsequently conclude with numerical examples which confirm the stated convergence rates. In the final chapter, we will conclude by discussing further potential extensions of the domain mapping method. Specifically, we will discuss how the domain mapping method may be applied to a Hele-Shaw problem and a two-phase Stefan problem, both posed over a random surface. In particular, we will discuss some of the challenges and possible approaches for these problems.

Chapter 2

The domain mapping method for elliptic equations posed over random curved domains

2.1 Introduction

In the mathematical characterisation of numerous engineering and biological systems, the topology of the domain can not be described precisely. The main sources of uncertainty are usually insufficient data, measurement errors or manufacturing variability, and inherent randomness in the model. This uncertainty in the geometry of the domain appears naturally in a broad range of applications, spanning from surface imaging [44], manufacturing of nano-devices [5], material science [13, 97, 102] and to even biological systems [7, 75]. As a result, the analysis of partial differential equations defined over random domains has become an interesting and rich mathematical field.

A comprehensive summary of some of the first directions for the analysis and numerical treatment of partial differential equations posed over random domains has been presented in [101]. Aside from the fictitious domain approach considered in [9, 80, 82, 83], the main approaches usually utilize a probabilistic framework by representing the random boundary of the domain, via a random field. This is either followed by the perturbation method [24, 48, 51] or the domain mapping method [10, 50, 101]. In the perturbation method, a shape Taylor expansion of the random boundary process is exploited to represent the solution of the random domain problem, and as a result, the method is limited to the consideration of random domains whose realisations only possess small perturbations around a fixed deterministic reference configuration. The domain mapping method, on the other hand does not suffer the same limitations. The key idea behind this approach is to instead construct an extension of the random boundary process into the interior of the random domain to form a complete stochastic domain mapping. The complete domain mapping defined over a deterministic reference domain, may subsequently be exploited to transform the initial partial differential equation posed over the random domain onto the deterministic reference domain, resulting in the consideration of a stochastic partial differential equation. For the reformulated problem, we may now refer to the extensive literature developed on the subject of the analysis and numerical treatment of stochastic partial differential equations posed over deterministic domains [28, 29, 47, 63, 66].

Our objective is to extend the applications of the domain mapping method to the consideration of random curved domains, such as random surfaces and random bulk-surface systems. As such, the analysis presented in this thesis will develop upon the well-established field of partial differential equations on stationary [18, 34, 42] and evolving surfaces [2, 31, 40]. While the original method is applicable to domains with random rough boundaries, we will limit our focus to sufficiently smooth random surfaces and leave the consideration of random rough surfaces for future work.

2.2 The domain mapping method

In this section, we shall provide an overview of the domain mapping method for differential equations in random domains, a method first introduced in [101] and subsequently analysed in [10, 46, 49, 50] and the references therein. As previously discussed, the method is based upon a parametric representation of the stochastic domain, constructed from an initial random field description of the boundary. We will therefore begin with a brief introduction on spaces of random fields. For further details on these spaces, we refer the reader to [66]. Note throughout this thesis, we will let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete, separable probability space consisting of a sample space Ω, a $\sigma-$algebra of events $\mathcal{F}$ and a probability measure $\mathbb{P}$.

2.2.1 Random field notation

For a given Banach space V and $p \in [1, \infty]$, the Lebesgue-Bochner space $L^p(\Omega; V)$ consists of all strongly $\mathcal{F}-$measurable functions $f : \Omega \to V$ for which the norm

$$\|f\|_{L^p(\Omega;V)} = \begin{cases} \left(\int_\Omega \|f(\omega)\|_V^p \, d\mathbb{P}(\omega)\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \operatorname*{ess\,sup}_\omega \|f(\omega)\|_V & p = \infty, \end{cases}$$

is finite. For convenience, we will express the parameters of a given random field $(f(\omega))(x)$ by $f(\omega, x)$. In the case that V is a separable Hilbert space, it follows that $L^2(\Omega; V)$ is also a separable Hilbert space and furthermore is isomorphic to the tensor product

$$L^2(\Omega; V) \cong L^2(\Omega) \otimes V. \tag{2.2.1}$$

For details, see [86].

2.2.2 The domain mapping method

To illustrate the key concepts of the domain mapping method, consider the following boundary value problem

$$\begin{aligned} -\Delta u(\omega) &= f(\omega) \quad \text{in } D(\omega) \\ u(\omega) &= 0 \quad \text{on } \Gamma(\omega), \end{aligned} \tag{2.2.2}$$

posed on an open, connected, bounded domain $D(\omega) \subset \mathbb{R}^2$ with a random boundary $\Gamma(\omega) = \partial D(\omega)$. Here the prescribed random field $f(\omega) : D(\omega) \to \mathbb{R}$ and additionally the boundary, will be assumed to be sufficiently regular to ensure well-posedness for *a.e.* ω. The first essential feature of the domain mapping method is the representation of the stochastic boundary via a random field. More precisely, in the above context we will assume that there exists a random field $\phi \in L^\infty(\Omega; L^\infty(\Gamma_0; \mathbb{R}^2))$, that maps a fixed closed curve $\Gamma_0 \subset \mathbb{R}^2$ onto realisations of the random boundary $\phi(\omega, \cdot) : \Gamma_0 \to \Gamma(\omega)$, see figure 2.1. Naturally, the subsequent analysis will require a greater regularity of the boundary process, however for the purpose of only illustrating the reformulation stage of the domain mapping method, we shall currently only assume boundedness. The next step in the method is to define an extension of the boundary process into the interior to form a stochastic mapping $\phi(\omega, \cdot) : \overline{D_0} \to \overline{D(\omega)}$ for the whole domain. For instance, [101] proposed an extension based on the solution of the Laplace equation over the unit square with boundary conditions prescribed by segments of the random boundary. However, alternative approaches may be considered depending on the application in question and the geometry of the computational reference domain.

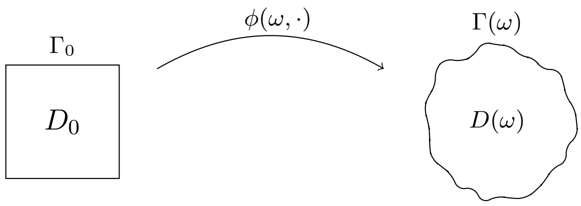

Figure 2.1: A realisation of the stochastic mapping.

With a complete domain mapping at hand, the random domain problem (2.2.2) can now be reformulated as a stochastic boundary value problem over the fixed deterministic domain D_0,

$$\begin{aligned} -\frac{1}{\sqrt{g(\omega)}} \nabla \cdot \left(\sqrt{g(\omega)} G^{-1}(\omega) \nabla (u \circ \phi)(\omega) \right) &= (f \circ \phi)(\omega) \quad \text{in } D_0 \\ u(\omega) &= 0 \quad \text{on } \Gamma_0, \end{aligned}$$

where the specific random coefficients for this particular problem are given by

$$G(\omega) = \nabla \phi^\top(\omega) \nabla \phi(\omega) \quad g(\omega) = \det G(\omega).$$

We now have access to a wide breadth of numerical techniques, including Monte-Carlo and the stochastic Galerkin method, to compute any statistical quantities of interest.

Remark 2.2.1.

1. *Note that the choice of the reference domain D_0, for the stochastic domain mapping ϕ describing the complete random geometry in question, is arbitrary and should be chosen in such a way that simplifies the computation at hand.*

2. *In practice, only statistical properties such as the expectation and two-point covariance function of the stochastic mapping ϕ will be known. As a result, an approximation of the true process will instead be used, commonly taking the form of a truncated series*

$$\phi(\omega, x) = \mathbb{E}[\phi](x) + \sum_{k=1}^{N} Y_k(\omega)\phi_k(x)$$

with centered, uncorrelated random coefficients Y_k with unit variance, such as a Karhunen-Loève expansion. Considerations of the induced error is beyond the scope of this thesis and we instead refer the reader to [49].

2.2.3 Quantity of interest

In order to give a precise definition of our quantity of interest, which for our purpose shall be some notion of a mean solution, we will first need to fix a suitable domain of definition. A natural choice would be the parametrisation based expected domain, introduced in [23] for random star-shaped domains, which we shall generalise as follows.

Definition 2.2.1 (Parametrisation based expected domain). *Given a family of random Lipschitz domains*

$$D(\omega) = \{\phi(\omega, x) \,|\, x \in D_0\} \subset \mathbb{R}^n, \tag{2.2.3}$$

parametrised over a fixed Lipschitz domain $D_0 \subset \mathbb{R}^n$ under the Lipschitz continuous mapping $\phi(\omega, \cdot) : D_0 \to \mathbb{R}^n$. Assuming $\phi(\cdot, x)$ is integrable for all $x \in D_0$, the parametrisation based expected domain $\mathbb{E}[D]$ of the random domain $D(\omega)$ is given by

$$\mathbb{E}[D] = \{\mathbb{E}[\phi](x) \,|\, x \in D_0\}. \tag{2.2.4}$$

Remark 2.2.2.

1. *Note that there are other alternative methods in which to define the expected value of a family of random sets. For example, we could characterise the random set $D(\omega)$ as an indicator function $1_{D(\omega)}$ and then use its average, the so-called coverage function $p(x) = \mathbb{P}(x \in D(\omega))$ to define the expected value to be set*

$$\mathbb{E}_V[D] = \{x \,|\, p(x) \geq \lambda\},$$

where the parameter $\lambda > 0$ is selected in a such a way that the volume of $\mathbb{E}_V[D]$ is close as possible to the expected volume of the random sets $D(\omega)$. This is known as the Vorob'ev expectation and was shown in [23] not to coincide with the parameterisation based expectation.

2. *Although there is no canonical definition of the expected value of a random domain, the parametrisation based expected domain fits naturally in the setting of the domain mapping method and thus will be adopted.*

Assumption 2.2.1. *We will assume that the expected value of the stochastic mapping*

$$\mathbb{E}[\phi] : D_0 \to \mathbb{E}[D],$$

is bi-Lipschitz continuous and furthermore that the boundary of the parametrisation-based expected domain $\mathbb{E}[D]$ is also Lipschitz continuous. This ensures that the expected domain is of the same dimension as D_0 and $D(\omega)$.

We will denote the induced zero-mean stochastic mapping between the parametrisation based expected domain $\mathbb{E}[D]$ and realisations of the random domain $D(\omega)$ by

$$\phi_e = \phi \circ \mathbb{E}[\phi]^{-1}. \tag{2.2.5}$$

See figure 2.2 for an illustration of the different mappings and domains. Our quantity of interest can now be defined on the expected domain as follows.

Definition 2.2.2 (QoI)**.** *Given a random field $u(\omega, \cdot) : D(\omega) \to \mathbb{R}$ defined over the family of random Lipschitz domains given in (2.2.3), the expected value of the random field is given by*

$$QoI = \mathbb{E}[u \circ \phi_e] \quad on\ \mathbb{E}[D]. \tag{2.2.6}$$

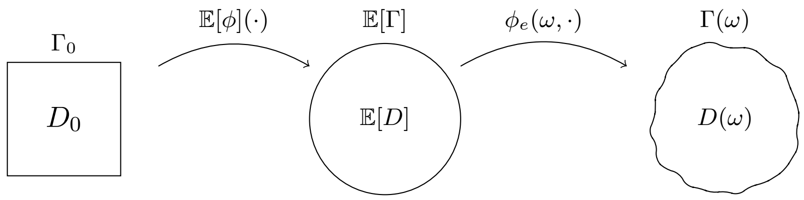

Figure 2.2: The computational domain, parametrisation based expected domain and a realisation of the random domain.

2.2.4 Extension to random surfaces

Our objective is the first consideration of the domain mapping method being applied to domains that involve random surfaces. While the method is applicable to domains with random rough boundaries, we will restrict our focus to random surfaces that are sufficiently smooth and will leave consideration of random, rough surfaces for future work. We will now proceed with some preliminary computations of geometric quantities as well as tangential derivatives of functions given over parametrised hypersurfaces in terms of quantites of the reference surface and derivatives of the domain mapping and corresponding pull-back function. This will provide a basis for the domain mapping method to be employed to several model PDEs over random surfaces.

2.3 Computations of tangential derivatives and geometric quanitities of parametrised hypersurfaces

Before we continue with the calculations, we will first introduce some notation for hypersurfaces that will be adopted throughout this thesis. For further details, see [26].

2.3.1 Hypersurface notation

A set $\Gamma \subset \mathbb{R}^{n+1}$ is said to be a C^k-hypersurface for $k \in \mathbb{N} \cup \{\infty\}$, provided that for every $x \in \Gamma$ there exists an open set $U \subset \mathbb{R}^{n+1}$ containing x and a smooth function $\varphi \in C^k(U)$ such that $\nabla\varphi(x) \neq 0$ on $U \cap \Gamma$ and

$$U \cap \Gamma = \{x \in U \,|\, \varphi(x) = 0\}. \tag{2.3.1}$$

A unit normal vector field to the hypersurface Γ can be computed via

$$\nu^\Gamma = \pm\frac{\nabla\varphi}{|\nabla\varphi|} \tag{2.3.2}$$

with the orientation chosen appropriately. For a differentiable function $f : \Gamma \to \mathbb{R}$, we define the tangential gradient of f

$$\nabla_\Gamma f = \nabla\bar{f} - (\nabla\bar{f} \cdot \nu^\Gamma)\nu^\Gamma = \mathcal{P}_\Gamma \nabla\bar{f} \tag{2.3.3}$$

where $\mathcal{P}_\Gamma = I - \nu^\Gamma \otimes \nu^\Gamma$ is the projection operator mapping onto the tangent space to Γ denoted $T\Gamma$ and $\bar{f}$ is a smooth extension of f to an open neighbourhood in $\mathbb{R}^{n+1}$. It can be shown that the tangential gradient is independent of the extension chosen, see Lemma 2.4 in [42] for details, and we shall denote its components by

$$\nabla_\Gamma f = (\underline{D}_1^\Gamma f, ..., \underline{D}_{n+1}^\Gamma f)^\top.$$

In addition, we define the Laplace-Beltrami operator for a twice differentiable function by

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f = \sum_{i=1}^{n+1} \underline{D}_i^\Gamma \underline{D}_i^\Gamma f. \tag{2.3.4}$$

We next introduce the Fermi coordinates with the following well-known lemma. These are a global coordinate system defined in an open neighbourhood around Γ in which every point can be uniquely expressed in terms of its signed distance $d^\Gamma(x)$ and its closest point $a^\Gamma(x)$ on the surface Γ .

Lemma 2.3.1. *Let d^Γ denote the signed distance function to Γ oriented in the chosen direction of the unit normal vector field ν^Γ. Then there exists $\delta > 0$ such that for every $x \in U_\delta := \{y \in \mathbb{R}^{n+1} \,|\, |d^\Gamma(y)| < \delta\}$ there exists a unique point $a^\Gamma(x) \in \Gamma$ that satisfies*

$$x = a^\Gamma(x) + d^\Gamma(x)\nu^\Gamma(a^\Gamma(x)). \tag{2.3.5}$$

Furthermore, assuming $\Gamma \in C^2$ it follows that $d^\Gamma \in C^2(U_\delta)$ and $a^\Gamma \in C^1(U_\delta)$ with

$$\nabla d^\Gamma(x) = \nu^\Gamma(a^\Gamma(x)) \tag{2.3.6}$$

$$\nabla a^\Gamma(x) = \left(I + d^\Gamma(x)\mathcal{H}^\Gamma(a^\Gamma(x))\right)^{-1} \mathcal{P}_\Gamma(a^\Gamma(x)). \tag{2.3.7}$$

Here $\mathcal{H}^\Gamma = \nabla_\Gamma \nu^\Gamma$ denotes the extended Weingarten map.

2.3.2 Considered geometric settings for parametrised surfaces

As a point of reference for the subsequent calculations, we will now describe the (deterministic) geometric settings considered for the parametrised surfaces. In all cases, the reference surface $\Gamma_0 \subset \mathbb{R}^{n+1}$ will be assumed to be of class at least C^2 and oriented by the unit normal vector field ν^{Γ_0}. The first geometric setting considered is that of a general parametrised surface.

Geometric setting 1 (Parametrised surface)**.** *The general parametrised hypersurface Γ will be given by*

$$\Gamma = \{\phi(x) \,|\, x \in \Gamma_0\} \subset \mathbb{R}^{n+1}, \tag{2.3.8}$$

for a given mapping $\phi : \Gamma_0 \to \mathbb{R}^{n+1}$.

We will later assume that the given surface parametrisation ϕ, is a sufficiently smooth diffeomorphism for the calculation in question. Furthermore, we shall denote the associated pull-back function of a given function $f : \Gamma \to \mathbb{R}$, onto the reference surface by

$$\hat{f} = f \circ \phi. \tag{2.3.9}$$

Motivated by many applications in which only small fluctuations of the random surface occur, we shall also consider the special case of a parametrised surface modelled as a graph over the reference surface.

Geometric setting 2 (Graphical surface)**.** *The graphical surface Γ, will be prescribed by*

$$\Gamma = \{\phi(x) = x + h(x)\nu^{\Gamma_0}(x) \,|\, x \in \Gamma_0\} \subset \mathbb{R}^{n+1} \tag{2.3.10}$$

for a given height function $h : \Gamma_0 \to \mathbb{R}$ defined over the reference surface.

The third and final geometric situation that we shall consider is when the parametrised surface is compact and encloses an open bulk domain.

Geometric setting 3 (Parametrised bulk-surface)**.** *The parametrised open bulk domain $D \subset \mathbb{R}^{n+1}$ and its boundary, the surface $\Gamma = \partial D$ will be given by*

$$D = \{\phi(x) \,|\, x \in D_0\} \qquad \Gamma = \{\phi(x) \,|\, x \in \Gamma_0\} \tag{2.3.11}$$

for a given parametrisation $\phi : \overline{D_0} \to \mathbb{R}^{n+1}$ defined over the open bulk domain $D_0 \subset \mathbb{R}^{n+1}$ with boundary $\Gamma_0 = \partial D_0$.

2.3.3 Tangential derivatives

Given the geometric setting of a general parametrised hypersurface described in (2.3.8), we have the following expressions for the pull-back of the tangential gradient and Laplace-Beltrami operator.

Lemma 2.3.2 (Tangential gradient)**.** *Given any differentiable function $f : \Gamma \to \mathbb{R}$, the pull-back of the tangential gradient is given by*

$$(\nabla_\Gamma f) \circ \phi = \left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right)^{-\top} \nabla_{\Gamma_0} \hat{f}. \tag{2.3.12}$$

Proof. Differentiating the associated pull-back function (2.3.9) and applying the chain rule gives

$$\nabla_{\Gamma_0} \hat{f} = \nabla_{\Gamma_0} \phi^\top \, (\nabla_\Gamma f) \circ \phi.$$

Since the tangential gradient of the surface parametrisation $\nabla_{\Gamma_0}\phi$ maps the tangent space $T_{(\cdot)}\Gamma_0$ into the tangent space $T_{\phi(\cdot)}\Gamma$ and additionally has kernel equal to $\mathrm{span}\{\nu^{\Gamma_0}\}$, we see that in order to invert the matrix $\nabla_{\Gamma_0}\phi$, we must first modify the corresponding linear map to bijectively map the space $\mathrm{span}\{\nu^{\Gamma_0}\}$ into $\mathrm{span}\{\nu^\Gamma \circ \phi\}$. One possible solution is to add the linear map $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ characterised by

$$L(\nu^{\Gamma_0}) = \nu^\Gamma \circ \phi, \quad L(\tau) = 0 \quad \tau \in T\Gamma_0,$$

which translates to adding the following tensor product

$$\nabla_{\Gamma_0} \hat{f} = \nabla_{\Gamma_0} \phi^\top \, (\nabla_\Gamma f) \circ \phi = \left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right)^\top (\nabla_\Gamma f) \circ \phi. \tag{2.3.13}$$

and thus leads to (2.3.12).

□

Remark 2.3.1. *The chain rule for tangential gradients (2.3.12) holds for any choice of orientation of the unit normals $\nu^{\Gamma_0}, \nu^\Gamma$ as a result of (2.3.13).*

Lemma 2.3.3 (Laplace-Beltrami operator)**.** *Given any $f : \Gamma \to \mathbb{R}$ twice differentiable, we have*

$$(\Delta_\Gamma f) \circ \phi = \frac{1}{\sqrt{g_{\Gamma_0}}} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0} \hat{f}\right), \tag{2.3.14}$$

where the coefficients are given by

$$G_{\Gamma_0} = \nabla_{\Gamma_0}\phi^\top \nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} \tag{2.3.15}$$

$$g_{\Gamma_0} = \det G_{\Gamma_0}. \tag{2.3.16}$$

Proof. Let us denote the given extension of the tangential gradient of the surface parametrisation appearing in (2.3.12) by $B = (b_{ij})_{i,j}$,

$$B = \nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0},$$

and furthermore denote its determinant by $b = \det B$ and the entries of its inverse by $B^{-1} = \left(b^{ij}\right)_{i,j}$. With the chain rule for tangential gradients (2.3.12), we can express the Laplace-Beltrami operator as follows

$$(\Delta_\Gamma f) \circ \phi = \sum_{i=1}^{n+1} (\underline{D}_i^\Gamma \underline{D}_i^\Gamma f) \circ \phi = \sum_{i,j=1}^{n+1} b^{ji} \underline{D}_j^{\Gamma_0} \left(\underline{D}_i^\Gamma f \circ \phi\right) = \sum_{i,j,k=1}^{n+1} b^{ji} \underline{D}_j^{\Gamma_0} \left(b^{ki} \underline{D}_k^{\Gamma_0} \hat{f}\right).$$

Writing in divergence form gives

$$\begin{aligned}(\Delta_\Gamma f) \circ \phi &= \sum_{i,j,k=1}^{n+1} \frac{1}{b} \underline{D}_j^{\Gamma_0} \left(b b^{ji} b^{ki} \underline{D}_k^{\Gamma_0} \hat{f}\right) - \sum_{i,j,k=1}^{n+1} \frac{1}{b} \underline{D}_j^{\Gamma_0} b\, b^{ji} b^{ki} \underline{D}_k^{\Gamma_0} \hat{f} - \sum_{i,j,k=1}^{n+1} \underline{D}_j^{\Gamma_0} b^{ji}\, b^{ki} \underline{D}_k^{\Gamma_0} \hat{f} \\ &= \frac{1}{b} \nabla_{\Gamma_0} \cdot \left(b B^{-1} B^{-\top} \nabla_{\Gamma_0} \hat{f}\right) + \mathrm{I} + \mathrm{II} \\ &= \frac{1}{\sqrt{g_{\Gamma_0}}} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0} \hat{f}\right) + \mathrm{I} + \mathrm{II}.\end{aligned}$$

The last step follows from the orthogonality result $\nabla_{\Gamma_0}\phi^\top (\nu^\Gamma \circ \phi) = 0$ by observing

$$B^\top B = \left(\nabla_{\Gamma_0}\phi^\top + \nu^{\Gamma_0} \otimes (\nu^\Gamma \circ \phi)\right) \left(\nabla_{\Gamma_0}\phi + (\nu^\Gamma \circ \phi) \otimes \nu^{\Gamma_0}\right) = \nabla_{\Gamma_0}\phi^\top \nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} = G_{\Gamma_0}$$

and consequently

$$b = \det(B) = \sqrt{\det(B^\top B)} = \sqrt{\det G_{\Gamma_0}} = \sqrt{g_{\Gamma_0}}.$$

We continue by proving that the remaining terms vanish. Recalling Jacobi's formula for the derivative of a determinant $\underline{D}_j^{\Gamma_0} \det B = \det B \operatorname{trace}\left(B^{-1} \underline{D}_j^{\Gamma_0} B\right)$ and computing the derivative of the inverse matrix $\underline{D}_j^{\Gamma_0} B^{-1} = -B^{-1} \underline{D}_j^{\Gamma_0} B B^{-1}$ gives

$$\frac{1}{b} \underline{D}_j^{\Gamma_0} b = \sum_{l,m=1}^{n+1} b^{lm} \underline{D}_j^{\Gamma_0} b_{ml}, \qquad \underline{D}_j^{\Gamma_0} b^{ji} = -\sum_{l,m=1}^{n+1} b^{jm} \underline{D}_j^{\Gamma_0} b_{ml} b^{li}.$$

It therefore follows after relabelling indices that

$$\begin{aligned} \mathrm{I}+\mathrm{II} &= -\sum_{i,j,k,l,m} b^{lm}\underline{D}_j^{\Gamma_0} b_{ml} b^{ji} b^{ki} \underline{D}_k^{\Gamma_0}\hat{f} + \sum_{i,j,k,l,m} b^{jm}\underline{D}_j^{\Gamma_0} b_{ml} b^{li} b^{ki}\underline{D}_k^{\Gamma_0}\hat{f} \\ &= \sum_{i,j,k,l,m} b^{lm}\left(\underline{D}_l^{\Gamma_0} b_{mj} - \underline{D}_j^{\Gamma_0} b_{ml}\right) b^{ji} b^{ki}\underline{D}_k^{\Gamma_0}\hat{f} \end{aligned} \tag{2.3.17}$$

Differentiating $b_{mj} := \underline{D}_j^{\Gamma_0}\phi_m + (\nu_m^\Gamma \circ \phi)\nu_j^{\Gamma_0}$ yields

$$\begin{aligned} \underline{D}_l^{\Gamma_0} b_{mj} - \underline{D}_j^{\Gamma_0} b_{ml} = {}& \underline{D}_l^{\Gamma_0}\underline{D}_j^{\Gamma_0}\phi_m - \underline{D}_j^{\Gamma_0}\underline{D}_l^{\Gamma_0}\phi_m + \underline{D}_l^{\Gamma_0}(\nu_m^\Gamma \circ \phi)\nu_j^{\Gamma_0} - \underline{D}_j^{\Gamma_0}(\nu_m^\Gamma \circ \phi)\nu_l^{\Gamma_0} \\ &+ (\nu_m^\Gamma \circ \phi)\underline{D}_l^{\Gamma_0}\nu_j^{\Gamma_0} - (\nu_m^\Gamma \circ \phi)\underline{D}_j^{\Gamma_0}\nu_l^{\Gamma_0}. \end{aligned}$$

By the symmetry of the Weingarten map $\underline{D}_l^{\Gamma_0}\nu_j^{\Gamma_0} = \underline{D}_j^{\Gamma_0}\nu_l^{\Gamma_0}$, we see that the last two terms cancel. We next interchange tangential derivatives with the formula

$$\underline{D}_l^{\Gamma_0}\underline{D}_j^{\Gamma_0}\phi_m - \underline{D}_j^{\Gamma_0}\underline{D}_l^{\Gamma_0}\phi_m = \left(\mathcal{H}^{\Gamma_0}\nabla_{\Gamma_0}\phi_m\right)_j \nu_l^{\Gamma_0} - \left(\mathcal{H}^{\Gamma_0}\nabla_{\Gamma_0}\phi_m\right)_l \nu_j^{\Gamma_0}$$

to obtain

$$\underline{D}_l^{\Gamma_0} b_{mj} - \underline{D}_j^{\Gamma_0} b_{ml} = \left(\underline{D}_l^{\Gamma_0}(\nu_m^\Gamma \circ \phi) - (\mathcal{H}^{\Gamma_0}\nabla_{\Gamma_0}\phi_m)_l\right)\nu_j^{\Gamma_0} + \left((\mathcal{H}^{\Gamma_0}\nabla_{\Gamma_0}\phi_m)_j - \underline{D}_j^{\Gamma_0}(\nu_m^\Gamma \circ \phi)\right)\nu_l^{\Gamma_0}.$$

Substituting into (2.3.17), we arrive at the following expression for the remaining terms

$$\begin{aligned} \mathrm{I}+\mathrm{II} = {}& \mathit{trace}\left(B^{-1}\nabla_{\Gamma_0}(\nu^\Gamma \circ \phi) - B^{-1}\nabla_{\Gamma_0}\phi\mathcal{H}^{\Gamma_0}\right) B^{-\top}\nu^{\Gamma_0} \cdot B^{-\top}\nabla_{\Gamma_0}\hat{f} \\ &+ \mathcal{H}^{\Gamma_0}\nabla_{\Gamma_0}\phi^\top B^{-\top}\nu^{\Gamma_0} \cdot B^{-1}B^{-\top}\nabla_{\Gamma_0}\hat{f} \\ &- \nabla_{\Gamma_0}(\nu^\Gamma \circ \phi)^\top B^{-\top}\nu^{\Gamma_0} \cdot B^{-1}B^{-\top}\nabla_{\Gamma_0}\hat{f}. \end{aligned}$$

Examining the first term, we have

$$B^{-\top}\nu^{\Gamma_0} \cdot B^{-\top}\nabla_{\Gamma_0}\hat{f} = B^{-1}B^{-\top}\nu^{\Gamma_0} \cdot \nabla_{\Gamma_0}\hat{f} = G_{\Gamma_0}^{-1}\nu^{\Gamma_0} \cdot \nabla_{\Gamma_0}\hat{f}.$$

Since $G_{\Gamma_0} = \nabla_{\Gamma_0}\phi^\top\nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}$ and thus $G_{\Gamma_0}^{-1}\nu^{\Gamma_0} = \nu^{\Gamma_0}$, the first term vanishes. For the second and third term, we observe that

$$B^{-\top}\nu^{\Gamma_0} = BB^{-1}B^{-\top}\nu^{\Gamma_0} = BG_{\Gamma_0}^{-1}\nu^{\Gamma_0} = B\nu^{\Gamma_0} = \nu^\Gamma \circ \phi.$$

Therefore as a result of the orthogonality results $\nabla_{\Gamma_0}\phi^\top(\nu^\Gamma \circ \phi) = 0$ and $\nabla_{\Gamma_0}(\nu^\Gamma \circ \phi)^\top(\nu^\Gamma \circ \phi) = 0$ which can be seen by

$$\underline{D}_i^{\Gamma_0}(\nu^\Gamma \circ \phi) \cdot (\nu^\Gamma \circ \phi) = \frac{1}{2}\underline{D}_i^\Gamma|\nu^\Gamma \circ \phi|^2 = 0,$$

we conclude $\mathrm{I}+\mathrm{II} = 0$. □

We will next compute the specific form of the coefficients for the Laplace-Beltrami operator given in Lemma 2.3.3 for the case of a graphical parametrisation over a reference surface.

Lemma 2.3.4 (Graphical case)**.** *Given a parametrised graphical surface* Γ*, as described in (2.3.10) where the surface parametrisation has the particular form*

$$\phi(x) = x + h(x)\nu^{\Gamma_0}(x) \quad x \in \Gamma_0, \tag{2.3.18}$$

for a given height function $h : \Gamma_0 \to \mathbb{R}$*, the coefficients (2.3.12), (2.3.14) of the Laplace-Beltrami operator on* Γ *simplify to give*

$$G_{\Gamma_0}^{-1} = A\left(I - \frac{A\nabla_{\Gamma_0}h \otimes A\nabla_{\Gamma_0}h}{1 + |A\nabla_{\Gamma_0}h|^2}\right)A \tag{2.3.19}$$

$$\sqrt{g_{\Gamma_0}} = \sqrt{1 + |A\nabla_{\Gamma_0}h|^2}\prod_{j=1}^{n}(1 + h\kappa_j^{\Gamma_0}). \tag{2.3.20}$$

Here $A := \left(I + h\mathcal{H}^{\Gamma_0}\right)^{-1}$ *and* $\{\kappa_j^{\Gamma_0}\}_j$ *denote the non-zero eigenvalues of the extended Weingarten map* $\mathcal{H}^{\Gamma_0}$*.*

Proof. Differentiating the surface parametrisation (2.3.18) gives

$$\nabla_{\Gamma_0}\phi = \mathcal{P}_{\Gamma_0} + h\mathcal{H}^{\Gamma_0} + \nu^{\Gamma_0} \otimes \nabla_{\Gamma_0}h.$$

Expanding G_{Γ_0} and cancelling orthogonal terms with the tensor product identity $(a\otimes b)(c\otimes d) = (b \cdot c)a \otimes d$, yields

$$\begin{aligned}
G_{\Gamma_0} &= \left(\mathcal{P}_{\Gamma_0} + h\mathcal{H}^{\Gamma_0} + \nabla_{\Gamma_0}h \otimes \nu^{\Gamma_0}\right)\left(\mathcal{P}_{\Gamma_0} + h\mathcal{H}^{\Gamma_0} + \nu^{\Gamma_0} \otimes \nabla_{\Gamma_0}h\right) + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} \\
&= \left(I + h\mathcal{H}^{\Gamma_0}\right)^2 + \nabla_{\Gamma_0}h \otimes \nabla_{\Gamma_0}h \\
&= A^{-1}\left(I + A\nabla_{\Gamma_0}h \otimes A\nabla_{\Gamma_0}h\right)A^{-1}.
\end{aligned}$$

Hence taking the inverse with the identity $(I + a \otimes b)^{-1} = I - \frac{a\otimes b}{1+a\cdot b}$ we obtain (2.3.19). For (2.3.20), we take the determinant using the identity $\det(I + a \otimes b) = 1 + a \cdot b$ which leads to

$$\det(G_{\Gamma_0}) = \left(1 + |A\nabla_{\Gamma_0}h|^2\right)\det(A^{-1})^2.$$

Observing that $A^{-1} = I + h\mathcal{H}^{\Gamma_0}$ has eigenvalues 1 and $\{1 + h\kappa_j^{\Gamma_0}\}_{j=1}^n$, we deduce

$$\det(A^{-1}) = \prod_{j=1}^{n}\left(1 + h\kappa_j^{\Gamma_0}\right)$$

and thus obtain the stated result for the surface area element $\sqrt{g_{\Gamma_0}} = \sqrt{detG_{\Gamma_0}}$. $\square$

2.3.4 The unit normal and Weingarten map

To obtain an expression for the unit normal vector field of a general parametrised hypersurface Γ, we begin by smoothly extending the given surface parametrisation to a C^1-diffeomorphic mapping $\bar{\phi} : U \to V$ between some open sets U and V containing Γ_0 and Γ respectively. The existence of such a mapping is guaranteed by the Whitney extension theorem [100]. We now have a level-set description of Γ

$$\Gamma = \{x \in V \mid d^{\Gamma_0}(\bar{\phi}^{-1}(x)) = 0\}$$

consequently leading to the following expression for a unit normal vector field due to (2.3.2).

Lemma 2.3.5 (Unit normal). *The pull-back of the unit normal vector field to the parametrised surface Γ described in (2.3.8), is given by*

$$\nu^{\Gamma} \circ \phi = \pm \frac{\nabla \bar{\phi}^{-\top} \nu^{\Gamma_0}}{|\nabla \bar{\phi}^{-\top} \nu^{\Gamma_0}|} \quad \textit{on } \Gamma_0. \tag{2.3.21}$$

Note that (2.3.21) can be shown to be independent of the extension chosen. As an example of a possible extension of the given surface parametrisation, consider the case of a graphical surface parametrisation as in (2.3.10).

Corollary 2.3.1 (Graphical case). *The unit normal vector field to a parametrised graphical surface (2.3.10) is given by*

$$\nu^{\Gamma} \circ \phi = \frac{\nu^{\Gamma_0} - A\nabla_{\Gamma_0} h}{|\nu^{\Gamma_0} - A\nabla_{\Gamma_0} h|} \tag{2.3.22}$$

where the orientation has been chosen to coincide with the reference surface Γ_0 when the height function is identically zero. Here $A = (1 + h\mathcal{H}^{\Gamma_0})^{-1}$.

Proof. We choose to extend the given surface parametrisation to small open neighbourhoods of both hypersurfaces contained in the open set in which the Fermi coordinates (2.3.5) are well defined, as follows

$$\bar{\phi}(x) = a^{\Gamma_0}(x) + \big(h(a^{\Gamma_0}(x)) + d^{\Gamma_0}(x)\big)\, \nu^{\Gamma_0}(a^{\Gamma_0}(x)).$$

The derivative of this extension evaluated on the reference surface Γ_0 simplifies with (2.3.6) and 2.3.7) to give

$$\nabla \bar{\phi} = I + h\mathcal{H}^{\Gamma_0} + \nu^{\Gamma_0} \otimes \nabla_{\Gamma_0} h = \big(I + \nu^{\Gamma_0} \otimes A\nabla_{\Gamma_0} h\big)\, A^{-1}.$$

Hence taking the inverse and applying the identity $(I + a \otimes b)^{-1} = 1 - \frac{a \otimes b}{1 + a \cdot b}$, recalling $A\nu^{\Gamma_0} = \nu^{\Gamma_0}$ we obtain

$$\nabla \bar{\phi}^{-\top} \nu^{\Gamma_0} = \big(I - \nu^{\Gamma_0} \otimes A\nabla_{\Gamma_0} h\big)\, A\nu^{\Gamma_0} = \nu^{\Gamma_0} - A\nabla_{\Gamma_0} h$$

and thus the stated result. Note that $\nu^{\Gamma_0} - A\nabla_{\Gamma_0} h \neq 0$ since the matrix $A = (1 + h\mathcal{H}^{\Gamma_0})$ maps the tangent space $T\Gamma$ into $T\Gamma$. □

We now compute the extended Weingarten map for the general parametrised hypersurface Γ with the help of the chain rule for tangential gradients (2.3.12) and the pull-back of the unit normal vector ν^{Γ} given in Lemma 2.3.5.

Lemma 2.3.6 (Extended Weingarten map). *Let the orientation of the parametrised hypersurface Γ described in (2.3.8), be fixed by a choice of a unit normal vector field ν^{Γ}. Then the* *pull-back of the extended Weingarten map is given by*

$$\mathcal{H}^\Gamma \circ \phi = -\left(\nabla_\Gamma \phi^{-\top} \circ \phi\right) C \left(\nabla_\Gamma \phi^{-1} \circ \phi\right) \tag{2.3.23}$$

where the matrix $C = (C_{ij})$ *is defined by*

$$C_{ij} = D_j^{\Gamma_0} D_i^{\Gamma_0} \phi \cdot (\nu^\Gamma \circ \phi). \tag{2.3.24}$$

Proof. Let us begin by extending the surface parametrisation in the normal direction as follows

$$\bar{\phi}(x) = \phi(a^{\Gamma_0}(x)) + d^{\Gamma_0}(x)\nu^\Gamma(\phi(a^{\Gamma_0}(x))).$$

Computing the derivative of the extension on the reference surface Γ_0 gives $\nabla\bar{\phi} = \nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}$ and thus with the expression (2.3.21) for the unit normal, leads to

$$\nu^\Gamma \circ \phi = \left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right)^{-\top} \nu^{\Gamma_0}.$$

We can now applying the chain rule for tangential gradients (2.3.12) with the matrix-vector identity $\nabla_{\Gamma_0}(Ab) = \left(D_j^{\Gamma_0} Ab \cdot e_i\right)_{ij} + A\nabla_{\Gamma_0} b$, to obtain

$$\begin{aligned}
\nabla_\Gamma \nu^\Gamma \circ \phi &= \nabla_{\Gamma_0}\left((\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}\nu^{\Gamma_0}\right)(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-1}\mathcal{P}_\Gamma \circ \phi \\
&= (\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}\mathcal{H}^{\Gamma_0}(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-1}\mathcal{P}_\Gamma \circ \phi \\
&+ B(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-1}\mathcal{P}_\Gamma \circ \phi
\end{aligned}$$

where $B = (B_{ij})$ is defined by $B_{ij} = D_j^{\Gamma_0}(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}\nu^{\Gamma_0} \cdot e_i$. Differentiating the inverse matrix yields

$$\begin{aligned}
B_{ij} &= -(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top} D_j^{\Gamma_0}(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^\top(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}\nu^{\Gamma_0} \cdot e_i \\
&= -(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top} D_j^{\Gamma_0}(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^\top \nu^\Gamma \circ \phi \cdot e_i.
\end{aligned}$$

Expanding and cancelling orthogonal terms gives

$$\begin{aligned}
&= -(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top} D_j^{\Gamma_0}\nabla_{\Gamma_0}\phi^\top \nu^\Gamma \circ \phi \cdot e_i - (\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top} D_j^{\Gamma_0}\nu^{\Gamma_0} \cdot e_i \\
&- \underbrace{\left(D_j^{\Gamma_0}(\nu^\Gamma \circ \phi) \cdot (\nu^\Gamma \circ \phi)\right)}_{=0}(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}\nu^{\Gamma_0} \cdot e_i.
\end{aligned}$$

Hence $B = -(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top}(C + \mathcal{H}^{\Gamma_0})$ with the matrix C defined as stated above and therefore it follows that

$$\mathcal{H}^\Gamma \circ \phi = -(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-\top} C(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0})^{-1}\mathcal{P}_\Gamma \circ \phi.$$

Finally, observing that $C : \mathbb{R}^{n+1} \to T\Gamma_0$ since $\left(C^\top \nu^{\Gamma_0}\right)_i = \sum_{j,k} \nu_j^{\Gamma_0} D_j^{\Gamma_0} D_i^{\Gamma_0}\phi_k(\nu_k^\Gamma \circ \phi) = 0$ and in addition

$$\left(\nabla_{\Gamma_0}\phi^\top + \nu^{\Gamma_0} \otimes \nu^\Gamma \circ \phi\right)^{-1} \mathcal{P}_{\Gamma_0} = \mathcal{P}_\Gamma \circ \phi \left(\nabla_{\Gamma_0}\phi^\top + \nu^{\Gamma_0} \otimes \nu^\Gamma \circ \phi\right)^{-1} = \nabla_\Gamma \phi^{-\top} \circ \phi,$$

we obtain the stated result. □

2.3.5 The normal derivative

We conclude this section by computing the pull-back of the normal derivative of functions defined over a parametrised bulk-surface as described in (2.3.11).

Lemma 2.3.7 (Normal derivative)**.** *Given any $u : \bar{D} \to \mathbb{R}$ sufficiently smooth, the pull-back of its normal derivative is given by*

$$\frac{\partial u}{\partial \nu_\Gamma} \circ \phi = \frac{\sqrt{g}}{\sqrt{g_{\Gamma_0}}} \left(\mathcal{P}_{\Gamma_0} G^{-1} \nu^{\Gamma_0} \cdot \nabla_{\Gamma_0} \hat{u} + \left(G^{-1}\nu^{\Gamma_0} \cdot \nu^{\Gamma_0}\right) \frac{\partial \hat{u}}{\partial \nu_{\Gamma_0}}\right) \tag{2.3.25}$$

where $G = \nabla\phi^\top \nabla\phi$ and $g = det(G)$ denoting its determinant.

Proof. Differentiating $u = \hat{u} \circ \phi^{-1}$ and substituting in the expression (2.3.21) for the unit normal ν^Γ where the orientation is chosen to be in the outer direction to the domain D gives

$$\frac{\partial u}{\partial \nu_\Gamma} = \nabla u \cdot \nu^\Gamma = \nabla\phi^{-\top}(\nabla\hat{u} \circ \phi^{-1}) \cdot \frac{\nabla\phi^{-\top}(\nu^{\Gamma_0} \circ \phi^{-1})}{|\nabla\phi^{-\top}(\nu^{\Gamma_0} \circ \phi^{-1})|}.$$

We next observe with the decomposition $\nabla\phi = \nabla_{\Gamma_0}\phi + \frac{\partial \phi}{\partial \nu_{\Gamma_0}} \otimes \nu^{\Gamma_0}$ and the orthogonality result $\nabla_{\Gamma_0}\phi^\top(\nu^\Gamma \circ \phi) = 0$ that

$$\nabla\phi^{-\top}\nu^{\Gamma_0} = \frac{1}{\frac{\partial \phi}{\partial \nu_{\Gamma_0}} \cdot (\nu^\Gamma \circ \phi)} \nu^\Gamma \circ \phi.$$

Since ϕ maps the boundary Γ_0 onto Γ, it follows that $\frac{\partial \phi}{\partial \nu_{\Gamma_0}} \cdot (\nu^\Gamma \circ \phi) > 0$ and thus

$$\frac{\partial u}{\partial \nu_\Gamma} \circ \phi = \left(\frac{\partial \phi}{\partial \nu_{\Gamma_0}} \cdot \nu^\Gamma \circ \phi\right) \nabla\hat{u} \cdot G^{-1}\nu^{\Gamma_0}.$$

We now continue by showing that the normal component of $\frac{\partial \phi}{\partial \nu_{\Gamma_0}}$ can be expressed as the ratio between the bulk $\sqrt{g}$ and the surface area element $\sqrt{g_{\Gamma_0}}$. This will be achieved in the context of exterior algebras.

Let $\tau_1, ..., \tau_n$ represent an orthonormal basis of the tangent space $T\Gamma_0$ and thus $\{\tau_1, ..., \tau_n, \nu^{\Gamma_0}\}$ forms a basis of $\mathbb{R}^{n+1}$. The determinant of linear map corresponding to $\nabla\phi$ evaluated on the boundary Γ_0 can be expressed in the notation of exterior algebras as follows

$$\det(\nabla\phi)\tau_1 \wedge ... \wedge \tau_n \wedge \nu^{\Gamma_0} = \nabla\phi\tau_1 \wedge ... \wedge \nabla\phi\tau_n \wedge \nabla\phi\nu^{\Gamma_0} = \nabla_{\Gamma_0}\phi\tau_1 \wedge ... \wedge \nabla_{\Gamma_0}\phi\tau_n \wedge \frac{\partial \phi}{\partial \nu_{\Gamma_0}}.$$

Since $\nabla_{\Gamma_0}\phi\tau_1, ..., \nabla_{\Gamma_0}\phi\tau_n$ form a basis of the tangent space $T\Gamma$ and the exterior product of any set of linearly dependent vectors is zero, we are therefore able to remove the tangent component of the normal derivative yielding

$$= \left(\frac{\partial\phi}{\partial\nu_{\Gamma_0}} \cdot (\nu^{\Gamma} \circ \phi)\right) \nabla_{\Gamma_0}\phi\tau_1 \wedge ... \wedge \nabla_{\Gamma_0}\phi\tau_n \wedge \nu^{\Gamma} \circ \phi.$$

Observing that each term in the above exterior product is the image of the basis $\{\tau_1, ..., \tau_n, \nu^{\Gamma_0}\}$ under the linear mapping $\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}$ gives

$$= \left(\frac{\partial\phi}{\partial\nu_{\Gamma_0}} \cdot (\nu^{\Gamma} \circ \phi)\right) \det\left(\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}\right) \tau_1 \wedge ... \wedge \tau_n \wedge \nu^{\Gamma_0}.$$

Hence it follows

$$\det\nabla\phi = \left(\frac{\partial\phi}{\partial\nu_{\Gamma_0}} \cdot \left(\nu^{\Gamma_0} \circ \phi\right)\right) \det\left(\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}\right).$$

We thus obtain the stated result with the following observations

$$\begin{aligned}
(\det\nabla\phi)^2 &= \det\nabla\phi^{\top}\nabla\phi = g \\
\left(\det\left(\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}\right)\right)^2 &= \det\left(\left(\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}\right)^{\top} \left(\nabla_{\Gamma_0}\phi + \left(\nu^{\Gamma} \circ \phi\right) \otimes \nu^{\Gamma_0}\right)\right) \\
&= \det\left(\nabla_{\Gamma_0}\phi^{\top}\nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}\right) = g_{\Gamma_0}.
\end{aligned}$$

□

2.4 First applications of the domain mapping method to random surfaces

We will now consider two model elliptic equations posed on the domains involving random surfaces. In particular, the first problem will be posed on a sufficiently smooth random surface and the second on a random bulk domain with a curved boundary. In both cases, the complete stochastic domain mapping will be assumed to be known. Employing the domain mapping method, we reformulate the equations onto their corresponding expected domain and prove well-posedness as well as a regularity result for the mean solution on the expected domain.

2.4.1 An elliptic equation on a random surface

Let $\Gamma(\omega)$ represent a random, compact C^2-hypersurface in $\mathbb{R}^{n+1}$ prescribed by

$$\Gamma(\omega) = \{\phi(\omega, x) \,|\, x \in \Gamma_0\} \tag{2.4.1}$$

for a given random field $\phi \in L^{\infty}(\Omega; C^2(\Gamma_0; \mathbb{R}^{n+1}))$ defined over a fixed, compact C^2-hypersurface $\Gamma_0 \subset \mathbb{R}^{n+1}$. We will assume that the stochastic mapping $\phi(\omega, \cdot) : \Gamma_0 \to \Gamma(\omega)$ is a C^2- diffeomorphism for almost every ω and furthermore satisfies the uniform bounds

$$\|\phi(\omega,\cdot)\|_{C^2(\Gamma_0)}, \|\phi^{-1}(\omega,\cdot)\|_{C^2(\Gamma(\omega))} < C \tag{2.4.2}$$

for some constant $C > 0$ independent of ω. We consider the following model elliptic equation on the random surface

$$-\Delta_{\Gamma(\omega)} u(\omega) + u(\omega) = f(\omega) \quad \text{on } \Gamma(\omega) \tag{2.4.3}$$

for a given random field $f(\omega,\cdot) : \Gamma(\omega) \to \mathbb{R}$. Our goal is to analyse the mean solution defined by

$$QoI := \mathbb{E}[u \circ \phi] \quad \text{on } \Gamma_0.$$

Reformulating (2.4.3) onto the expected domain with the calculation of the Laplace-Beltrami operator provided in Lemma 2.3.2 yields

$$-\frac{1}{\sqrt{g_{\Gamma_0}(\omega)}} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega) \nabla_{\Gamma_0} \hat{u}(\omega) \right) + \hat{u}(\omega) = \hat{f}(\omega) \quad \text{on } \Gamma_0, \tag{2.4.4}$$

where the random coefficient is given by

$$G_{\Gamma_0}(\omega) = \nabla_{\Gamma_0} \phi^\top(\omega) \nabla_{\Gamma_0} \phi(\omega) + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} \tag{2.4.5}$$

with $g_{\Gamma_0}(\omega) = \det G_{\Gamma_0}(\omega)$. Multiplying through by surface area element $\sqrt{g_{\Gamma_0}(\omega)}$ and integrating by parts, we arrive at the following mean-weak formulation.

Problem 2.4.1 (Mean-weak formulation)**.** *Given $\hat{f} \in L^2(\Omega; L^2(\Gamma_0))$, find $\hat{u} \in L^2(\Omega; H^1(\Gamma_0))$ such that*

$$\int_\Omega \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{u}(\omega) \cdot \nabla_{\Gamma_0} \hat{\varphi}(\omega) + \hat{u}(\omega)\hat{\varphi}(\omega)\sqrt{g_{\Gamma_0}(\omega)} = \int_\Omega \int_{\Gamma_0} \hat{f}(\omega)\hat{\varphi}(\omega)\sqrt{g_{\Gamma_0}(\omega)} \tag{2.4.6}$$

for every $\hat{\varphi} \in L^2(\Omega; H^1(\Gamma_0))$. Here, we have set $\mathcal{D}_{\Gamma_0}(\omega) = \sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega)$.

We denote the associated bilinear form $a(\cdot,\cdot) : L^2(\Omega; H^1(\Gamma_0)) \times L^2(\Omega; H^1(\Gamma_0)) \to \mathbb{R}$ and linear functional $l(\cdot) : L^2(\Omega; L^2(\Gamma_0)) \to \mathbb{R}$ by

$$a(\hat{u}, \hat{\varphi}) = \int_\Omega \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{u}(\omega) \cdot \nabla_{\Gamma_0} \hat{\varphi}(\omega) + \hat{u}(\omega)\hat{\varphi}(\omega)\sqrt{g_{\Gamma_0}(\omega)} \tag{2.4.7}$$

$$l(\hat{\varphi}) = \int_\Omega \int_{\Gamma_0} \hat{f}(\omega)\hat{\varphi}(\omega)\sqrt{g_{\Gamma_0}(\omega)}. \tag{2.4.8}$$

Thus the mean-weak formulation can be written more succiently as

$$a(\hat{u}, \hat{\varphi}) = l(\hat{\varphi}) \quad \text{for all } \hat{\varphi} \in L^2(\Omega; H^1(\Gamma_0)). \tag{2.4.9}$$

Proposition 2.4.1. *Under the uniformity assumptions (2.4.2) on the stochastic mapping, there exists a constants $C_{D_{\Gamma_0}}, C_{g_{\Gamma_0}} > 0$ such that the singular values σ_i of $\mathcal{D}_{\Gamma_0}$ and the surface area* *element* $\sqrt{g_{\Gamma_0}}$ *are bounded above and below by*

$$0 < C_{D_{\Gamma_0}}^{-1} \leq \sigma_i\left(\mathcal{D}_{\Gamma_0}(\omega, x)\right) \leq C_{D_{\Gamma_0}} < +\infty \tag{2.4.10}$$

$$0 < C_{g_{\Gamma_0}}^{-1} \leq \sqrt{g_{\Gamma_0}(\omega, x)} \leq C_{g_{\Gamma_0}} < +\infty \tag{2.4.11}$$

for all $x \in \Gamma_0$ *and a.e.* ω.

Proof. We can rewrite G_{Γ_0} using the orthogonality $\nabla_{\Gamma_0}\phi^\top(\nu^\Gamma \circ \phi) = 0$, as follows

$$G_{\Gamma_0} = \nabla_{\Gamma_0}\phi^\top \nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} = \left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right)^\top \left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right).$$

Examining each term separately, we see that the inverse is given by

$$\left(\nabla_{\Gamma_0}\phi + \nu^\Gamma \circ \phi \otimes \nu^{\Gamma_0}\right)^{-1} = \nabla_\Gamma \phi^{-1} + \nu^{\Gamma_0} \otimes \nu^\Gamma \circ \phi.$$

Hence it follows

$$\begin{aligned} G_{\Gamma_0}^{-1} &= \left(\nabla_\Gamma\phi^{-1} + \nu^{\Gamma_0} \otimes \nu^\Gamma \circ \phi\right)\left(\nabla_\Gamma\phi^{-1} + \nu^{\Gamma_0} \otimes \nu^\Gamma \circ \phi\right) \\ &= \left(\nabla_\Gamma\phi^{-1} \circ \phi\right)\left(\nabla_\Gamma\phi^{-\top} \circ \phi\right) + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}. \end{aligned}$$

Therefore with (2.4.2), we have uniform bounds above and below on the singular values of $G_{\Gamma_0}(\omega)$ and hence obtain the estimates (2.4.10) and (2.4.11). ☐

A direct consequence of the above uniform bounds on the random coefficients is the existence and uniqueness of a solution to (2.4.6) guaranteed by the Lax-Milgram theorem.

Theorem 2.4.1. *Given any* $\hat{f} \in L^2(\Omega; L^2(\Gamma_0))$, *there exists a unique solution* $\hat{u}$ *to the mean-weak formulation (2.4.6) that satisfies the energy estimate*

$$\|\hat{u}\|_{L^2(\Omega;H^1(\Gamma_0))} \leq c\|\hat{f}\|_{L^2(\Omega;L^2(\Gamma_0))}. \tag{2.4.12}$$

Proof. The stability estimate (2.4.12) follows from the coercivity of $a(\cdot,\cdot)$. ☐

By considering the original surface equation (2.4.3) on $\Gamma(\omega) \in C^2$, we would expect from standard elliptic surface regularity results that for given $f(\omega) \in L^2(\Gamma(\omega))$, the pathwise solution belongs to $u(\omega) \in H^2(\Gamma(\omega))$ and therefore $\hat{u}(\omega) \in H^2(\Gamma_0)$ for a.e. ω. However since the H^2 a-priori estimate on $u(\omega)$ will naturally depend on the geometry of the realisation $\Gamma(\omega)$, it is not immediately clear whether the solution to the mean-weak formulation belongs to $\hat{u} \in L^2(\Omega; H^2(\Gamma_0))$. We will therefore continue by explicitly treating all arising constants and their dependency on the geometry of the random domain.

Theorem 2.4.2 (Regularity)**.** *Given any* $\hat{f} \in L^2(\Omega; L^2(\Gamma_0))$, *the solution to (2.4.6) belongs to* $\hat{u} \in L^2(\Omega; H^2(\Gamma_0))$ *and furthermore satisfies the following estimate*

$$\|\hat{u}\|_{L^2(\Omega;H^2(\Gamma_0))} \leq C\|\hat{f}\|_{L^2(\Omega;L^2(\Gamma_0))}. \tag{2.4.13}$$

Proof. Let us consider the push-forward $u = \hat{u} \circ \phi^{-1}$ of realisations of the weak solution onto $\Gamma(\omega)$ for almost every ω, which as a result of the tensor structure $L^2(\Omega; H^1(\Gamma_0)) \cong L^2(\Omega) \otimes H^1(\Gamma_0)$ are weak solutions of

$$-\Delta_{\Gamma(\omega)} u(\omega) + u(\omega) = f(\omega) \quad \text{on } \Gamma(\omega) \tag{2.4.14}$$

with $f = \hat{f} \circ \phi^{-1}$. Since for almost every $\omega \in \Omega$, $\Gamma(\omega)$ is C^2 and $f(\omega) \in L^2(\Gamma(\omega))$, it follows that $u(\omega) \in H^2(\Gamma(\omega))$ and therefore $\hat{u}(\omega) \in H^2(\Gamma_0)$. For the a-priori estimate (2.4.13), it was shown in [42] through a series of integration by parts and interchanging of tangential derivatives that the H^2 semi-norm satisfies

$$|u(\omega)|_{H^2(\Gamma(\omega))} \leq \|\Delta_{\Gamma(\omega)} u(\omega)\|_{L^2(\Gamma(\omega))} + c(\omega)|u(\omega)|_{H^1(\Gamma(\omega))}$$

with

$$c(\omega) = \sqrt{\|H^{\Gamma(\omega)}\mathcal{H}^{\Gamma(\omega)} - 2\left(\mathcal{H}^{\Gamma(\omega)}\right)^2\|_{L^\infty(\Gamma(\omega))}}.$$

Here $H^{\Gamma(\omega)} = trace\left(\mathcal{H}^{\Gamma(\omega)}\right)$ is the mean-curvature. Hence with the uniform bounds (2.4.2) on the stochastic mapping and the previously calculated expression (2.3.23) for the Weingarten map, we obtain an upper bound on the constant $c(\omega)$ independent of ω. Thus, with the PDE (2.4.14) pointwise we have the bound

$$\|u(\omega)\|_{H^2(\Gamma(\omega))} \leq c\left(\|f(\omega)\|_{L^2(\Gamma(\omega))} + \|u(\omega)\|_{H^1(\Gamma(\omega))}\right).$$

We can now pull-back onto the expected domain, applying the norm equivalence of the pull-back transformation

$$C^{-1}\|\hat{u}(\omega)\|_{H^k(\Gamma_0)} \leq \|u(\omega)\|_{H^k(\Gamma(\omega))} \leq C\|\hat{u}(\omega)\|_{H^k(\Gamma_0)} \quad \text{for } k = 0, 1, 2 \text{ and a.e. } \omega$$

where the constants are independent of ω due to bounds (2.4.2), and the stability estimate (2.4.12) to obtain

$$\|\hat{u}(\omega)\|_{H^2(\Gamma_0)} \leq C\|\hat{f}(\omega)\|_{L^2(\Gamma_0)}.$$

and thus the stated result. □

2.4.2 A coupled elliptic system on a random bulk-surface

For the second problem, we consider a coupled elliptic system on a random bulk-surface motivated by the deterministic case analysed in [34]. More precisely, the geometric setting is as follows. We let $\{\Gamma(\omega)\}$ denote a family of random, compact C^2-hypersurfaces in $\mathbb{R}^{n+1}$ enclosing open domains $D(\omega)$ and will denote the outer unit normal by $\nu^{\Gamma(\omega)}$. The family of random domains will be prescribed by the stochastic mapping

$$\phi : \overline{D_0} \to \overline{D(\omega)} \quad \phi|_{\Gamma_0} : \Gamma_0 \to \Gamma(\omega), \tag{2.4.15}$$

where the reference surface $\Gamma_0 \subset \mathbb{R}^{n+1}$ will also be a compact C^2-hypersurface with open interior D_0. We will assume that the domain mapping is a C^2-diffeomorphism for *a.e.* $\omega \in \Omega$ and additionally satisfies

$$\|\phi(\omega,\cdot)\|_{C^2(\overline{D_0})}, \quad \|\phi^{-1}(\omega,\cdot)\|_{C^2(\overline{D(\omega)})} < C \tag{2.4.16}$$

for a constant $C > 0$ independent of ω. The proposed coupled elliptic system on the random bulk-surface reads as follows

$$-\Delta u(\omega) + u(\omega) = f(\omega) \quad \text{on } D(\omega) \tag{2.4.17a}$$

$$\alpha u(\omega) - \beta v(\omega) + \frac{\partial u}{\partial \nu_\Gamma}(\omega) = 0 \quad \text{on } \Gamma(\omega) \tag{2.4.17b}$$

$$-\Delta_{\Gamma(\omega)} v(\omega) + v(\omega) + \frac{\partial u}{\partial \nu_\Gamma}(\omega) = f_\Gamma(\omega) \quad \text{on } \Gamma(\omega). \tag{2.4.17c}$$

Here $\alpha, \beta > 0$ are given positive constants and $f(\omega,\cdot) : D(\omega) \to \mathbb{R}$ and $f_\Gamma(\omega,\cdot) : \Gamma(\omega) \to \mathbb{R}$ are prescribed random fields. As with our previous problem, our quantity of interest is the mean solution, that is the pair $(\mathbb{E}[u], \mathbb{E}[v])$ defined by

$$\mathbb{E}[u] := \mathbb{E}[u \circ \phi] \quad \mathbb{E}[v] := \mathbb{E}[v \circ \phi].$$

Let us continue by reformulating the system (2.4.17) onto the expected domain $\overline{D_0}$ with our previously calculated expressions for the Laplace-Beltrami operator (2.3.14) and the normal derivative (2.3.25) giving

$$-\frac{1}{\sqrt{g(\omega)}} \nabla \cdot \left(\sqrt{g(\omega)} G^{-1}(\omega) \nabla \hat{u}(\omega) \right) + \hat{u}(\omega) = \hat{f}(\omega) \quad \text{in } D_0 \tag{2.4.18a}$$

$$\alpha \hat{u}(\omega) - \beta \hat{v}(\omega) + \frac{\sqrt{g(\omega)}}{\sqrt{g_{\Gamma_0}(\omega)}} G^{-1}(\omega) \nu^{\Gamma_0} \cdot \nabla \hat{u}(\omega) = 0 \quad \text{on } \Gamma_0 \tag{2.4.18b}$$

$$-\frac{1}{\sqrt{g_{\Gamma_0}(\omega)}} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega) \right) + \hat{v}(\omega) + \frac{\sqrt{g(\omega)}}{\sqrt{g_{\Gamma_0}(\omega)}} G^{-1}(\omega) \nu^{\Gamma_0} \cdot \nabla \hat{u} = \hat{f}_{\Gamma_0}(\omega) \quad \text{on } \Gamma_0. \tag{2.4.18c}$$

Here the random coefficients are

$$G(\omega) = \nabla \phi^\top(\omega) \nabla \phi(\omega) \quad G_{\Gamma_0}(\omega) = \nabla_{\Gamma_0} \phi^\top(\omega) \nabla_{\Gamma_0} \phi(\omega) + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}$$

with $g(\omega) = detG(\omega)$, $g_{\Gamma_0}(\omega) = detG_{\Gamma_0}(\omega)$. For convenience, we have set $\hat{f}_{\Gamma_0} = f_\Gamma \circ \phi$. To derive a mean-weak formulation, we follow the varitional approach presented in [34]. We begin by multiplying through the bulk equation (2.4.18a) by the area element $\sqrt{g}$ and integrating by parts which gives

$$\int_{D_0} \sqrt{g(\omega)} G^{-1}(\omega) \nabla \hat{u}(\omega) \cdot \nabla \hat{\varphi}(\omega) + \hat{u}(\omega)\hat{\varphi}(\omega)\sqrt{g(\omega)} \tag{2.4.19}$$
$$- \int_{\Gamma_0} \left(\sqrt{g(\omega)} G^{-1}(\omega) \nabla \hat{u}(\omega) \cdot \nu^{\Gamma_0} \right) \hat{\varphi}(\omega). = \int_{D_0} \hat{f}(\omega)\hat{\varphi}(\omega)\sqrt{g(\omega)}$$

Similarly, for the surface equation (2.4.18c) we integrate by parts recalling that the hypersurface Γ_0 is without boundary, to obtain

$$\int_{\Gamma_0} \sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega) \cdot \nabla_{\Gamma_0} \hat{\xi} + \hat{v}(\omega)\hat{\xi}(\omega)\sqrt{g_{\Gamma_0}(\omega)} \tag{2.4.20}$$
$$+ \int_{\Gamma_0} \sqrt{g(\omega)} \left(G^{-1}(\omega) \nu^{\Gamma_0} \cdot \nabla \hat{u}(\omega) \right) \hat{\xi}(\omega) = \int_{\Gamma_0} \hat{f}_{\Gamma_0}(\omega)\hat{\xi}(\omega)\sqrt{g_{\Gamma_0}(\omega)}.$$

Taking the weighted sum and substituting in the reformulated Robin boundary condition (2.4.18b), we arrive at the following mean-weak formulation:

Problem 2.4.2 (Mean-weak formulation)**.** *Given any $\hat{f} \in L^2(\Omega; L^2(D_0))$ and $\hat{f}_{\Gamma_0} \in L^2(\Omega; L^2(\Gamma_0))$, find $\hat{u} \in L^2(\omega; H^1(D_0))$ and $\hat{v} \in L^2(\Omega; H^1(\Gamma_0))$ such that*

$$\begin{aligned} &\alpha \int_\Omega \int_{D_0} \mathcal{D}(\omega) \nabla \hat{u}(\omega) \cdot \nabla \hat{\varphi}(\omega) + \hat{u}(\omega)\hat{\varphi}(\omega)\sqrt{g(\omega)} \\ &+\beta \int_\Omega \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega) \cdot \nabla_{\Gamma_0} \hat{\xi}(\omega) + \hat{v}(\omega)\hat{\xi}(\omega)\sqrt{g_{\Gamma_0}(\omega)} \\ &+ \int_\Omega \int_{\Gamma_0} (\alpha \hat{u}(\omega) - \beta \hat{v}(\omega))(\alpha \hat{\varphi}(\omega) - \beta \hat{\xi}(\omega))\sqrt{g_{\Gamma_0}(\omega)} \\ &\qquad\qquad = \alpha \int_\Omega \int_{D_0} \hat{f}(\omega)\hat{\varphi}(\omega)\sqrt{g(\omega)} + \beta \int_\Omega \int_{\Gamma_0} \hat{f}_{\Gamma_0}(\omega)\hat{\xi}(\omega)\sqrt{g_{\Gamma_0}(\omega)}. \end{aligned}$$

for every $\hat{\varphi} \in L^2(\Omega; H^1(D_0))$ and $\hat{\xi} \in L^2(\Omega; H^1(\Gamma_0))$. Here we set $\mathcal{D}_{\Gamma_0}(\omega) = \sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega)$.

We denote the associated bilinear form and linear functional stated above by

$$a(\cdot,\cdot) : L^2(\Omega; V) \times L^2(\Omega; V) \to \mathbb{R}, \quad l(\cdot) : L^2(\Omega; H) \to \mathbb{R}, \tag{2.4.21}$$

where we have set $H = L^2(D_0) \times L^2(\Gamma_0)$ and $V = H^1(D_0) \times H^1(\Gamma_0)$ to be Hilbert spaces equipped with respective inner products

$$\begin{aligned} ((\hat{u},\hat{v}),(\hat{\varphi},\hat{\xi}))_H &= (\hat{u},\hat{\varphi})_{L^2(D_0)} + (\hat{v},\hat{\xi})_{L^2(\Gamma_0)}, \\ ((\hat{u},\hat{v}),(\hat{\varphi},\hat{\xi}))_V &= (\hat{u},\hat{\varphi})_{H^1(D_0)} + (\hat{v},\hat{\xi})_{H^1(\Gamma_0)}. \end{aligned}$$

The mean-weak formulation thus reads as follows

$$a((\hat{u},\hat{v}),(\hat{\varphi},\hat{\xi})) = l((\hat{\varphi},\hat{\xi})). \tag{2.4.22}$$

The following uniform bounds on the random bulk coefficients follow immediately from the assumption (2.4.16) on the stochastic mapping. Furthermore, the derived bounds on the surface coefficients presented in Proposition 2.4.1 also hold since the tangential derivatives of the surface parametrisation and its inverse are also uniformly bounded as a consequence of (2.4.16).

Proposition 2.4.2 (Uniform bounds). *There exists constants $C_g, C_D > 0$ such that the bulk area element $\sqrt{g(\omega)}$ and the singular values σ_i of $D(\omega)$ are uniformly bounded for all $x \in D_0$ and a.e. ω by*

$$0 < C_g^{-1} \leq \sqrt{g(\omega, x)} \leq C_g < +\infty \tag{2.4.23}$$

$$0 < C_D^{-1} \leq \sigma_i \left(D(\omega, x)\right) \leq C_D < +\infty. \tag{2.4.24}$$

We will now establish the existence and uniqueness of a solution to the mean-weak formulation (2.4.22) as well as a regularity result, where we shall develop upon the results derived in [34] for a specific deterministic case.

Theorem 2.4.3. *Given any $(\hat{f}, \hat{f}_{\Gamma_0}) \in H$, there exist a unique solution $(\hat{u}, \hat{v}) \in L^2(\Omega; V)$ to (2.4.22) which satisfies the energy estimate*

$$\|(\hat{u}, \hat{v})\|_{L^2(\Omega;V)} \leq c\|(\hat{f}, \hat{f}_{\Gamma_0})\|_{L^2(\Omega;H)}. \tag{2.4.25}$$

Proof. With our uniform bounds (2.4.23), (2.4.10) on the random bulk and surface coefficients, we can now proceed in verifying all the conditions of the Lax-Milgram theorem are satisified. For a coercivity estimate, we argue

$$\begin{aligned}
a((\hat{u}, \hat{v}), (\hat{u}, \hat{v})) &\geq \alpha \min\left(C_D^{-1}, C_g^{-1}\right) \|\hat{u}\|^2_{L^2(\Omega;H^1(D_0))} + \beta \min\left(C_{D_{\Gamma_0}}^{-1}, C_{g_{\Gamma_0}}^{-1}\right) \|\hat{v}\|^2_{L^2(\Omega;H^1(\Gamma_0))} \\
&\quad + C_{g_{\Gamma_0}}^{-1} \|\alpha\hat{u} - \beta\hat{v}\|^2_{L^2(\Omega;L^2(\Gamma_0))} \\
&\geq C(\|\hat{u}\|^2_{L^2(\Omega;H^1(D_0))} + \|\hat{v}\|^2_{L^2(\Omega;H^1(\Gamma_0))}) \\
&= C\|(\hat{u}, \hat{v})\|^2_{L^2(\Omega;V)}.
\end{aligned}$$

For the continuity of the bilinear form $a(\cdot, \cdot)$, we apply the Cauchy-Schwarz inequality with the boundedness of the trace operator $\|f\|_{L^2(\Gamma_0)} \leq c_T\|f\|_{H^1(D_0)}$ as follows

$$\begin{aligned}
&|a((\hat{u}, \hat{v}), (\hat{\varphi}, \hat{\xi}))| \\
\leq &\alpha \max(C_D, C_g)\|\hat{u}\|_{L^2(\Omega;H^1(D_0))}\|\hat{\varphi}\|_{L^2(\Omega;H^1(D_0))} + \beta \max(C_{D_{\Gamma_0}}, C_{g_{\Gamma_0}})\|\hat{v}\|_{L^2(\Omega;H^1(\Gamma_0))}\|\hat{\xi}\|_{L^2(\Omega;H^1(\Gamma_0))} \\
&\quad + C_{g_{\Gamma_0}}\|\alpha\hat{u} - \beta\hat{v}\|_{L^2(\Omega;L^2(\Gamma_0))}\|\alpha\hat{\varphi} - \beta\hat{\xi}\|_{L^2(\Omega;L^2(\Gamma_0))} \\
\leq &C\|(\hat{u}, \hat{v})\|_{L^2(\Omega;V)}\|(\hat{\varphi}, \hat{\xi})\|_{L^2(\Omega;V)} \\
&\quad + C_{g_{\Gamma_0}} \left(\alpha c_T\|\hat{u}\|_{L^2(\Omega;H^1(D_0))} + \beta\|\hat{v}\|_{L^2(\Omega;L^2(\Gamma_0))}\right) \left(c_T\|\hat{\varphi}\|_{L^2(\Omega;H^1(D_0))} + \|\hat{\xi}\|_{L^2(\Omega;L^2(\Gamma_0))}\right) \\
\leq &C\|(\hat{u}, \hat{v})\|_{L^2(\Omega;V)}\|(\hat{\varphi}, \hat{\xi})\|_{L^2(\Omega;V)}.
\end{aligned}$$

Thus we have the existence and uniqueness of a solution to (2.4.22). The estimate (2.4.25) then follows from coercivity of $a(\cdot, \cdot)$. □

Theorem 2.4.4 (Regularity)**.** *Given any $\hat{f} \in L^2(\Omega; L^2(D_0))$ and $\hat{f}_{\Gamma_0} \in L^2(\Omega; L^2(\Gamma_0))$, the mean-weak solution $(\hat{u}, \hat{v})$ to (2.4.22) satisfies*

$$\hat{u} \in L^2(\Omega; H^2(D_0)) \quad \hat{v} \in L^2(\Omega; H^2(\Gamma_0)). \tag{2.4.26}$$

Furthermore, we have

$$\|(\hat{u}, \hat{v})\|_{L^2(\Omega; H^2(D_0) \times H^2(\Gamma_0))} \leq C \|(\hat{f}, \hat{f}_{\Gamma_0})\|_{L^2(\Omega; L^2(D_0) \times L^2(\Gamma_0))}, \tag{2.4.27}$$

where the constant $C > 0$ depends only the geometry of the reference domain $\overline{D_0}$ and the uniform bound (2.4.16) on the stochastic mapping.

Proof. Observe that for *a.e.* $\omega \in \Omega$, the solution $(\hat{u}, \hat{v})$ satisfies for every $\hat{\varphi} \in H^1(D_0)$ and $\hat{\xi} \in H^1(\Gamma_0)$,

$$\alpha \int_{D_0} \mathcal{D}(\omega) \nabla \hat{u}(\omega) \cdot \nabla \hat{\varphi} + \hat{u}(\omega) \hat{\varphi} \sqrt{g(\omega)} + \beta \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega) \cdot \nabla_{\Gamma_0} \hat{\xi} + \hat{v}(\omega) \hat{\xi} \sqrt{g_{\Gamma_0}(\omega)}$$
$$+ \int_{\Gamma_0} \left(\alpha \hat{u}(\omega) - \beta \hat{v}(\omega)\right) \left(\alpha \hat{\varphi} - \beta \hat{\xi}\right) \sqrt{g_{\Gamma_0}(\omega)} = \alpha \int_{D_0} \hat{f}(\omega) \hat{\varphi} \sqrt{g(\omega)} + \beta \int_{\Gamma_0} \hat{f}_{\Gamma_0}(\omega) \hat{\xi} \sqrt{g_{\Gamma_0}(\omega)}.$$

Setting $\hat{\varphi} = 0$ gives

$$\beta \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega) \cdot \nabla_{\Gamma_0} \hat{\xi} + \hat{v}(\omega) \hat{\xi} \sqrt{g_{\Gamma_0}(\omega)}$$
$$- \int_{\Gamma_0} \left(\alpha \hat{u}(\omega) - \beta \hat{v}(\omega)\right) \beta \hat{\xi} \sqrt{g_{\Gamma_0}(\omega)} = \beta \int_{\Gamma_0} \hat{f}_{\Gamma_0}(\omega) \hat{\xi} \sqrt{g_{\Gamma_0}(\omega)}.$$

Hence we see that $\hat{v}(\omega)$ is the pathwise weak solution to the elliptic surface equation

$$-\beta \nabla_{\Gamma_0} \cdot \left(\mathcal{D}_{\Gamma_0}(\omega) \nabla_{\Gamma_0} \hat{v}(\omega)\right) + (\beta + \beta^2) \sqrt{g_{\Gamma_0}(\omega)} \hat{v}(\omega) = \alpha\beta \sqrt{g_{\Gamma_0}(\omega)} \hat{u}(\omega) + \beta \sqrt{g_{\Gamma_0}(\omega)} \hat{f}_{\Gamma_0}(\omega).$$

It therefore follows form the surface regularity result given in Theorem 2.4.2 since $\hat{u}(\omega) \in L^2(\Gamma_0)$, that $\hat{v}(\omega) \in H^2(\Gamma_0)$ for a.e. ω and furthermore

$$\|\hat{v}\|_{H^2(\Gamma_0)} \leq C \left(\|\hat{f}_{\Gamma_0}(\omega)\|_{L^2(\Gamma_0)} + \|\hat{u}(\omega)\|_{L^2(\Gamma_0)} \right) \tag{2.4.28}$$

where the constant $C > 0$ is independent of ω. To obtain higher regularity of the bulk quantity, we set $\hat{\xi} = 0$ yielding

$$\alpha \int_{D_0} \mathcal{D}(\omega) \nabla \hat{u}(\omega) \cdot \nabla \hat{\varphi} + \hat{u}(\omega) \hat{\varphi} \sqrt{g(\omega)} + \int_{\Gamma_0} \left(\alpha \hat{u}(\omega) - \beta \hat{v}(\omega)\right) \alpha \hat{\varphi} \sqrt{g_{\Gamma_0}(\omega)} = \alpha \int_{D_0} \hat{f}(\omega) \hat{\varphi} \sqrt{g(\omega)}.$$

This is precisely the weak formulation of the following elliptic boundary value problem subject to the reformulated Robin boundary condition

$$\begin{aligned}-\alpha\nabla\cdot(\mathcal{D}(\omega)\nabla\hat{u}(\omega))+\alpha\sqrt{g(\omega)}\hat{u}(\omega)&=\alpha\sqrt{g(\omega)}\hat{f}(\omega)\quad\text{in }D_0\\ \mathcal{D}(\omega)\nabla\hat{u}(\omega)\cdot\nu^{\Gamma_0}+\alpha\sqrt{g_{\Gamma_0}(\omega)}\hat{u}(\omega)&=\beta\sqrt{g_{\Gamma_0}(\omega)}\hat{v}(\omega)\quad\text{on }\Gamma_0.\end{aligned}$$

Since the coefficients are sufficiently regular, more precisely

$$\begin{aligned}&\mathcal{D}_{ij}(\omega)\in C^1\left(\overline{D}_0\right),\quad \alpha\sqrt{g(\omega)}\in L^\infty(D_0),\quad \alpha\sqrt{g(\omega)}\hat{f}(\omega)\in L^2(D_0),\\ &0<\alpha_0\le\alpha\sqrt{g_{\Gamma_0}(\omega)}\in C^1(\Gamma_0),\quad \beta\sqrt{g_{\Gamma_0}(\omega)}\hat{v}(\omega)\in H^1(\Gamma_0),\end{aligned}$$

and the boundary is sufficiently smooth $\Gamma_0\in C^2$, we can apply standard regularity results [61] to deduce $\hat{u}(\omega)\in H^2(D_0)$ for a.e. ω with the estimate

$$\|\hat{u}(\omega)\|_{H^2(D_0)}\le C\left(\|\hat{f}(\omega)\|_{L^2(D_0)}+\|\hat{v}(\omega)\|_{H^1(\Gamma_0)}\right).\tag{2.4.29}$$

Here the constant $C>0$ is independent of ω since all the coefficients are uniformly bounded and furthermore, $\mathcal{D}(\omega)$ is uniformly elliptic in ω. Combining (2.4.28) and (2.4.29) with the stability estimate (2.4.25) and boundedness of the trace operator leads to

$$\begin{aligned}\|\hat{u}(\omega)\|_{H^2(D_0)}+\|\hat{v}(\omega)\|_{H^2(\Gamma_0)}&\le C\left(\|\hat{f}_{\Gamma_0}(\omega)\|_{L^2(\Gamma_0)}+C_T\|\hat{u}(\omega)\|_{H^1(\Gamma_0)}+\|\hat{f}(\omega)\|_{L^2(D_0)}+\|\hat{v}(\omega)\|_{H^1(\Gamma_0)}\right)\\ &\le C\left(\|\hat{f}_{\Gamma_0}(\omega)\|_{L^2(\Gamma_0)}+\|\hat{f}(\omega)\|_{L^2(D_0)}\right).\end{aligned}$$

and hence the stated result. □

2.5 An abstract analysis of elliptic equations on random curved domains

We continue by considering in an abstract setting, the mean-weak formulation of general elliptic equations on random curved domains after being transformed onto the expected domain via the given stochastic domain mapping. Working in this abstract framework, we will present and analyse a finite element discretisation coupled with the Monte-Carlo method to approximate our quantity of interest, the mean solution. As the expected domain is assumed to be curved, the proposed finite element method will involve perturbations of the variational set up corresponding to the approximation of the geometric domain. An optimal error bound in the energy norm for our non-conforming approach is derived with the help of the first lemma of Strang with suitable assumptions on the finite element space approximation and arising consistency error. Furthermore, an $L^2(\Omega;L^2)$-type estimate is proved by a standard duality argument.

2.5.1 Abstract mean-weak formulation

Let V and H denote separable Hilbert spaces for which the embedding $V\hookrightarrow H$ is dense and continuous. We assume that we are in the setting where we have a sample dependent bilinear form $\tilde{a}(\omega;\cdot,\cdot) : V \times V \to \mathbb{R}$ and linear functional $\tilde{l}(\omega;\cdot) : H \to \mathbb{R}$ corresponding to the path-wise weak formulation

$$\tilde{a}(\omega; u(\omega), \varphi) = \tilde{l}(\omega; \varphi)$$

of the elliptic equation after being reformulated onto the expected domain. For convenience, we will omit the pull-back notation for functions $\hat{u}$ since all the subsequent analysis will be considered on the expected domain. The mean-weak formulation will thus in general read as follows:

Problem 2.5.1 (Mean-weak formulation)**.** *Find $u \in L^2(\Omega; V)$ such that for every $\varphi \in L^2(\Omega; V)$ we have*

$$\int_\Omega \tilde{a}(\omega; u(\omega), \varphi(\omega))\, d\mathbb{P}(\omega) = \int_\Omega \tilde{l}(\omega; \varphi(\omega))\, d\mathbb{P}(\omega). \tag{2.5.1}$$

We denote the associated bilinear form $a(\cdot,\cdot) : L^2(\Omega; V) \times L^2(\Omega; V) \to \mathbb{R}$ and linear functional $l(\cdot) : L^2(\Omega; H) \to \mathbb{R}$ by

$$a(u, \varphi) = \int_\Omega \tilde{a}(\omega; u(\omega), \varphi(\omega)), \quad l(\varphi) = \int_\Omega \tilde{l}(\omega, \varphi(\omega)).$$

and shall assume all the requirements of the Lax-Milgram theorem are satisfied thus ensuring the existence and uniqueness of the solution.

2.5.2 Abstract formulation of the finite element discretisation

For a given $h \in (0, h_0)$, let $\mathcal{V}_h$ be a finite dimensional space that will represent a finite element space and let V_h and H_h denote the space $\mathcal{V}_h$ endowed with respective norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{H_h}$. We assume that V_h and H_h are Hilbert spaces and furthermore that $V_h \hookrightarrow H_h$ is uniformly embedded, that is

$$\|\chi_h\|_{H_h} \leq c\|\chi_h\|_{V_h} \quad \text{for all } \chi_h \in V_h,$$

for a constant $c > 0$ independent of h. In practice, the spaces V_h and H_h will represent equivalent Hilbert spaces to the continuous solution spaces V and H but posed over a discrete approximation of the curved domain, with h denoting the discretisation parameter. We introduce the sample-dependent bilinear form and linear functional

$$\tilde{a}_h(\omega;\cdot,\cdot) : \mathcal{V}_h \times \mathcal{V}_h \to \mathbb{R} \quad \tilde{l}_h(\omega;\cdot) : \mathcal{V}_h \to \mathbb{R},$$

that are perturbations approximating their continuous counterparts and will assume $\tilde{a}_h(\omega : \cdot, \cdot)$ is uniformly V_h-elliptic and bounded and additionally $\tilde{l}_h(\omega;\cdot)$ is uniformly bounded. More precisely, there exists constants $c_1, c_2, c_3 > 0$ independent of ω and h such that

$$\tilde{a}_h(\omega; \chi_h, \chi_h) \geq c_1\|\chi\|^2_{V_h} \tag{2.5.2}$$
$$|\tilde{a}_h(\omega; \chi_h, W_h)| \leq c_2\|\chi_h\|_{V_h}\|W_h\|_{V_h} \tag{2.5.3}$$
$$|\tilde{l}(\omega; \chi_h)| \leq c_3\|\chi_h\|_{H_h}. \tag{2.5.4}$$

The finite element approximation of the mean-weak formulation (2.5.1) for a given finite dimensional subspace $\mathcal{V}_h \subset V_h$ will then take the following form:

Problem 2.5.2 (Semi-discrete problem)**.** *Find $U_h \in L^2(\Omega; \mathcal{V}_h)$ such that*

$$a_h(U_h, \phi_h) = \int_\Omega \tilde{a}_h(\omega; U_h(\omega), \phi_h(\omega))\, d\mathbb{P}(\omega) = \int_\Omega \tilde{l}_h(\omega; \phi_h(\omega))\, d\mathbb{P}(\omega) = l_h(\phi_h) \tag{2.5.5}$$

for all $\phi_h \in L^2(\Omega; \mathcal{V}_h)$.

By our uniform assumptions of the bilinear form $\tilde{a}(\omega; \cdot, \cdot)$ and the linear functional $\tilde{l}(\omega; \cdot)$, we deduce the existence and uniqueness of a solution to the semi-discrete problem.

Theorem 2.5.1. *There exists a unique solution $U_h \in L^2(\Omega; V_h)$ to the semi-discrete problem (2.5.5) that satisfies*

$$\|U_h\|_{L^2(\Omega; V_h)} \leq C \tag{2.5.6}$$

with the constant $C > 0$ is independent of $h \in (0, h_0)$.

Observe that if we let $\{\chi_j\}_{j=1}^N$ be a basis of $\mathcal{V}_h$ and express $U_h, \phi_h \in L^2(\Omega; \mathcal{V}_h) \cong L^2(\Omega) \otimes \mathcal{V}_h$ in the form

$$U_h(\omega) = \sum_{j=1}^N U_j(\omega)\chi_j \quad \phi_h(\omega) = \sum_{j=1}^N \phi_j(\omega)\chi_j,$$

where $U(\omega) = (U_1(\omega), ..., U_N(\omega))^\top \in L^2(\Omega)^N$ and $\Phi(\omega) = (\phi_1(\omega), ..., \phi_N(\omega))^\top \in L^2(\Omega)^N$, then (2.5.5) can be rewritten as

$$\int_\Omega \Phi(\omega) \cdot S(\omega) U(\omega) = \int_\Omega \Phi(\omega) \cdot F(\omega). \tag{2.5.7}$$

Here the random stiffness matrix $S(\omega) = (S_{ij}(\omega))_{i,j=1,...,N}$ and load vector $F(\omega) = (F_j(\omega))_{j=1,...,N}$ are given by $S_{ij}(\omega) = \tilde{a}_h(\omega; \chi_j, \chi_i)$, $F_j(\omega) = \tilde{l}_h(\omega; \chi_j)$. Since $\phi_j(\omega) \in L^2(\Omega)$ are arbitrary, we deduce that the semi-discrete problem is equivalent to finding $U \in L^2(\Omega; \mathbb{R}^N)$ which satisfies

$$S(\omega)U(\omega) = F(\omega) \quad \text{for a.e. } \omega. \tag{2.5.8}$$

2.5.3 Assumptions on the finite element approximation and the continuous equations

We now state all the necessary assumptions that will be required in deriving an error estimate for the semi-discrete solution. In order to compare our semi-discrete solution with the continuous solution, we first need to assume the existence of a lifting map.

Assumption 2.5.1 (Lifting map)**.** *There exists a linear mapping $\Lambda_h : \mathcal{V}_h \to V$ for which there* *exists constants* $c_1, c_2 > 0$ *independent of* $h \in (0, h_0)$ *such that for all* $\chi_h \in \mathcal{V}_h$

$$c_1 \|\chi_h\|_{H_h} \le \|\Lambda_h \chi_h\|_H \le c_2 \|\chi_h\|_{H_h} \tag{L1}$$

$$c_1 \|\chi_h\|_{V_h} \le \|\Lambda_h \chi_h\|_V \le c_2 \|\chi_h\|_{V_h}. \tag{L2}$$

We denote the lifted finite dimensional space by $V_h^l := \Lambda_h \mathcal{V}_h$. Next, we introduce the Hilbert space $Z_0 \hookrightarrow V$ which shall represent a space consisting of functions of higher regularity for which we assume we have the following interpolation estimate.

Assumption 2.5.2 (Approximation of finite element space)**.** *There exists a well-defined interpolation operator* $I_h : Z_0 \to V_h^l$ *for which there exists* $c > 0$ *such that*

$$\|\eta - I_h \eta\|_H + h\|\eta - I_h \eta\|_V \le ch^2 \|\eta\|_{Z_0} \quad \textit{for } \eta \in Z_0. \tag{I1}$$

Naturally, the lifting map and interpolation operator can be extended to random functions in a pathwise sense

$$(\Lambda_h \phi_h)(\omega) := \Lambda_h \phi_h(\omega) \quad (I_h \phi_h)(\omega) := I_h \phi_h(\omega),$$

and the previous estimates (L1),(L2), (I1) hold for their respective norms $\|\cdot\|_{L^2(\Omega;H)}$ and $\|\cdot\|_{L^2(\Omega;V)}$. We continue by imposing bounds on the consistency error arising from the pertubation of the variational form. For this, we will assume the existence of an inverse lifting map $\Lambda_h : L^2(\Omega; Z_0) \to L^2(\Omega; V_h)$ and will denote inverse lift of a function w by w^{-l}.

Assumption 2.5.3 (Consistency error)**.** *Given any* $W_h, \phi_h \in L^2(\Omega; \mathcal{V}_h)$ *with corresponding lifts* $w_h, \chi_h \in L^2(\Omega; V_h^l)$*, we have the bounds*

$$|l(\varphi_h) - l_h(\phi_h)| \le ch^2 \|\varphi_h\|_{L^2(\Omega;H)} \tag{P1}$$

$$|a(w_h, \varphi_h) - a_h(W_h, \phi_h)| \le ch\|w_h\|_V \|\varphi_h\|_{L^2(\Omega;V)}. \tag{P2}$$

Furthermore, for any $w, \varphi \in L^2(\Omega; Z_0)$ *with inverse lifts* w^{-l}, φ^{-l} *we have*

$$|a(w, \varphi) - a_h(w^{-l}, \varphi^{-l})| \le ch^2 \|w\|_{L^2(\Omega;Z_0)} \|\varphi\|_{L^2(\Omega;Z_0)}. \tag{P3}$$

Our final assumption will be on the regularity of an associated dual problem that will enable us to derive an $L^2(\Omega; H)$ error estimate using the standard Aubin-Nitsche trick. The associated dual problem reads as follows:

Problem 2.5.3 (Dual problem)**.** *For a given* $g \in L^2(\Omega; H)$*, find* $w(g) \in L^2(\Omega; V)$ *such that*

$$a(\varphi, w(g)) = (g, \varphi)_{L^2(\Omega;H)} \quad \textit{for } \varphi \in L^2(\Omega; V). \tag{2.5.9}$$

Here $(\cdot, \cdot)_{L^2(\Omega;H)}$ denotes the inner product on the Hilbert space $L^2(\Omega; H)$.

Assumption 2.5.4 (Regularity of dual problem). *The solution $w(g)$ to the dual problem belongs to space $L^2(\Omega; Z_0)$ and furthermore satisfies*

$$\|w(g)\|_{L^2(\Omega;Z_0)} \leq c\|g\|_{L^2(\Omega;H)} \tag{R1}$$

for a constant $c > 0$ independent of both g and $h \in (0, h_0)$.

2.5.4 Error estimates for the semi-discrete solution

Recall that the abstract finite element space $\mathcal{V}_h$ is not necessarily contained in the Hilbert space V. However, with the assumed existence of a lifting map

$$\Lambda_h : L^2(\Omega; \mathcal{V}_h) \to L^2(\Omega; V_h^l) \subset L^2(\Omega; V),$$

we can lift the discrete bilinear form $a_h(\cdot, \cdot)$ and the linear functional $l_h(\cdot)$ onto the space $L^2(\Omega; V_h^l)$ by the following relations for $w_h = \Lambda_h W_h, \varphi_h = \Lambda_h \phi_h \in L^2(\Omega; V_h^l)$

$$a_h^l(w_h, \varphi_h) : = a_h(W_h, \phi_h) \quad l_h^l(w_h) := l_h(W_h), \tag{2.5.10}$$

thus inducing a third variational problem equivalent to (2.5.5).

Problem 2.5.4 (Lifted semi-discrete problem). *Find $u_h \in L^2(\Omega; V_h^l)$ such that for every $\varphi_h \in L^2(\Omega; V_h^l)$ we have*

$$a_h^l(u_h, \varphi_h) = l_h^l(\varphi_h). \tag{2.5.11}$$

Since $L^2(\Omega; V_h^l)$ is contained in the solution space $L^2(\Omega; V)$, the lifted semi-discrete problem fits into the abstract non-conforming finite element setting considered in the first lemma of Strang [94]. We will now present these results in the context of our random Hilbert space setting.

Lemma 2.5.1 (First lemma of Strang). *Let u_h denote the solution to the lifted semi-discrete problem (2.5.11) and assume that the bilinear form $a_h^l(\cdot, \cdot)$ is uniformly $L^2(\Omega; V_h^l)$-elliptic, i.e. for some $\alpha > 0$*

$$a_h^l(\varphi_h, \varphi_h) \geq \alpha\|\varphi\|^2_{L^2(\Omega;V)}$$

for all $\varphi \in L^2(\Omega; V_h^l)$ and $h \in (0, h_0)$. Then there exists a constant $C > 0$ independent of h such that

$$\|u - u_h\|_{L^2(\Omega;V)} \lesssim \inf_{\varphi_h \in L^2(\Omega;V_h^l)} \left(\|u - \varphi_h\|_{L^2(\Omega;V)} + \sup_{w_h \in L^2(\Omega;V_h^l)} \frac{|a(\varphi_h, w_h) - a_h^l(\varphi_h, w_h)|}{\|w_h\|_{L^2(\Omega;V)}} \right) \tag{2.5.12}$$
$$+ \sup_{w_h \in L^2(\Omega;V_h^l)} \frac{|l(w_h) - l_h^l(w_h)|}{\|w_h\|_{L^2(\Omega;V)}}.$$

Theorem 2.5.2 (Error estimates). *Let u denote the solution of the continuous problem (2.5.1) and assume that it is sufficiently regular $u \in L^2(\Omega; Z_0)$ and let U_h be the discrete solution of (2.5.5) with lift $u_h = \Lambda_h U_h$. Then with the assumptions listed in section 2.5.3 satisfied, there exists a constant $c > 0$ such that for all $h \in (0, h_0)$ we have the error estimate*

$$\|u - u_h\|_{L^2(\Omega;H)} + h\|u - u_h\|_{L^2(\Omega;V)} \leq ch^2\|u\|_{L^2(\Omega;Z_0)}. \tag{2.5.13}$$

Proof. It follows from the uniform ellipticity assumption (2.5.2) on the bilinear form $a_h(\cdot,\cdot)$ and the norm equivalence of the lifting map, that for any $\varphi_h = \Lambda_h \phi_h \in L^2(\Omega; V_h^l)$ we have

$$a_h^l(\varphi_h, \varphi_h) = a_h(\phi_h, \phi_h) \geq c\|\phi_h\|^2_{L^2(\Omega;V_h)} \geq c\|\varphi_h\|^2_{L^2(\Omega;V)}.$$

Therefore the bilinear form $a_h^l(\cdot,\cdot)$ is uniformly coercive and thus we can apply the first lemma of Strang. Substituting $\varphi_h = I_h u$ into the estimate (2.5.12) and inserting the consistency bounds (P1), (P2) gives

$$\|u - u_h\|_{L^2(\Omega;V)} \lesssim \|u - I_h u\|_{L^2(\Omega;V)} + h\|I_h\|_{L^2(\Omega;V)} + h^2.$$

Hence with the interpolation estimate (I1) applied to $u \in L^2(\Omega; Z_0)$ we obtain

$$\|u - u_h\|_{L^2(\Omega;V)} \lesssim h\|u\|_{L^2(\Omega;Z_0)}. \tag{2.5.14}$$

For the $L^2(\Omega; H)-$estimate, we use a standard duality argument. Given $g \in L^2(\Omega; H)$ and an arbitrary $w_h \in L^2(\Omega; V_h^l)$ we have

$$\begin{aligned}
(u - u_h, g)_{L^2(\Omega;H)} &= a(u - u_h, w(g) - w_h) + a(u - u_h, w_h) \\
&= a(u - u_h, w(g) - w_h) + l(w_h) - l_h^l(w_h) - \Big(a(u_h, w_h) - a_h^l(u_h, w_h)\Big) \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Choosing $w_h = I_h w(g)$ and applying the interpolation estimate (I1) to the solution of the dual problem which is assumed (R1) to be sufficiently regular $w(g) \in L^2(\Omega; Z_0)$ gives

$$\begin{aligned}
|\mathrm{I}| &\lesssim \|u - u_h\|_{L^2(\Omega;V)}\|w(g) - I_h w(g)\|_{L^2(\Omega;V)} \\
&\lesssim h^2\|u\|_{L^2(\Omega;Z_0)}\|w(g)\|_{L^2(\Omega;Z_0)} \\
&\lesssim h^2\|u\|_{L^2(\Omega;Z_0)}\|g\|_{L^2(\Omega;H)}.
\end{aligned}$$

We bound the consistency error in the second term with (P2) giving

$$|\mathrm{II}| \lesssim h^2\|I_h w(g)\|_{L^2(\Omega;V)} \lesssim h^2\|w(g)\|_{L^2(\Omega;Z_0)} \lesssim h^2\|g\|_{L^2(\Omega;H)}.$$

To obtain a bound of order h^2 for the third term, we begin by rewriting it as follows

$$\begin{aligned}\mathrm{III} &= a(u_h, w(g) - I_h w(g)) - a_h^l(u_h, w(g) - I_h w(g)) \\ &\quad + a(u - u_h, w(g)) - a_h^l(u - u_h, w(g)) \\ &\quad - \Big(a(u, w(g)) - a_h^l(u, w(g))\Big).\end{aligned}$$

Now we are able to apply the estimate (P3) to the last term since both $u, w(g) \in L^2(\Omega; Z_0)$ and can then follow a similar argument as to the previous cases for the first two terms which leads to

$$\begin{aligned}|\mathrm{III}| &\lesssim h\|u_h\|_{L^2(\Omega;V)}\|w(g) - I_h w(g)\|_{L^2(\Omega;V)} + h\|u - u_h\|_{L^2(\Omega;V)}\|w(g)\|_{L^2(\Omega;V)} \\ &\quad + h^2\|u\|_{L^2(\Omega;Z_0)}\|w(g)\|_{L^2(\Omega;Z_0)} \\ &\lesssim h^2\|w(g)\|_{L^2(\Omega;Z_0)} + h^2\|u\|_{L^2(\Omega;Z_0)}\|w(g)\|_{L^2(\Omega;Z_0)} \\ &\lesssim h^2\|u\|_{L^2(\Omega;Z_0)}\|g\|_{L^2(\Omega;H)}.\end{aligned}$$

Combining the results gives the stated result

$$\|u - u_h\|_{L^2(\Omega;H)} = \sup_{g \in L^2(\Omega;H)\setminus\{0\}} \frac{(u - u_h, g)_{L^2(\Omega;H)}}{\|g\|_{L^2(\Omega;H)}} \lesssim h^2\|u\|_{L^2(\Omega;Z_0)}.$$

□

We conclude our abstract error analysis by combining our finite element discretisation with the Monte-Carlo method to estimate our quantity of interest, the mean solution $E[u]$. Recall, that for an arbitrary Hilbert space $\mathcal{H}$, the Monte-Carlo estimator of the expectation of a random variable $Y \in L^2(\Omega; \mathcal{H})$ is a $\mathcal{H}$-valued random variable $E_M[Y] : \otimes_{i=1}^M \Omega \to \mathcal{H}$ defined by

$$E_M[Y] = \frac{1}{M}\sum_{i=1}^M \hat{Y}_i$$

where $M \in \mathbb{N}$ is the chosen number of samples taken and $\hat{Y}_i$ are independent identically distributed copies of the random variable Y. Furthermore, we have the following well-known convergence result, see [66].

Lemma 2.5.2 (Monte-Carlo convergence rate)**.** *For a given $M \in \mathbb{N}$ and a $\mathcal{H}$-valued random variable $Y \in L^2(\Omega; \mathcal{H})$, the Monte-Carlo estimator satisfies the convergence rate*

$$\|E[Y] - E_M[Y]\|_{L^2(\Omega^M;\mathcal{H})} \le \frac{1}{\sqrt{M}}\|Y\|_{L^2(\Omega;\mathcal{H})}. \tag{2.5.15}$$

Therefore, if we consider the error between the mean solution $\mathbb{E}[u]$ and our discrete approximation $\mathbb{E}[u_h]$ in the $L^2(\Omega^M; H)$ norm, and decompose it into the error arising from the finite element discretisation and the statistical error for the Monte-Carlo approximation, we obtain the following bound

$$\begin{aligned}
\|E[u]-E_M[u_h]\|_{L^2(\Omega^M;H)} &\leq \|E[u]-E[u_h]\|_{L^2(\Omega^M;H)}+\|E[u_h]-E_M[u_h]\|_{L^2(\Omega^M;H)}\\
&\leq \|u-u_h\|_{L^2(\Omega;H)}+\frac{1}{\sqrt{M}}\|u_h\|_{L^2(\Omega;H)} \lesssim h^2+\frac{1}{\sqrt{M}}
\end{aligned}$$

A similar argument in the $L^2(\Omega;V)$ leads to the following convergence rates.

Theorem 2.5.3. *Let all the conditions from Theorem 2.5.2 be satisfied. Then we have the following error estimates*

$$\|E[u]-E_M[u_h]\|_{L^2(\Omega^M;H)} \lesssim h^2+\frac{1}{\sqrt{M}} \tag{2.5.16}$$

$$\|E[u]-E_M[u_h]\|_{L^2(\Omega^M;V)} \lesssim h+\frac{1}{\sqrt{M}}. \tag{2.5.17}$$

2.6 Discretisation of the reformulated elliptic PDEs on their expected domains

In this section, we apply the results from the abstract theory to two finite element discretisation schemes for the reformulations of the two model elliptic equations. In each case, we will verify that all the listed assumptions in abstract setting are satisfied hence giving the stated convergence rate.

2.6.1 The elliptic equation on a random surface

To discretise the reformulated elliptic surface equation

$$-\Delta_{\Gamma(\omega)}u(\omega)+u(\omega)=f(\omega) \quad \text{on } \Gamma(\omega)$$

on the expected domain, we propose a semi-discrete scheme using linear Lagrangian surface finite elements, as first introduced in [42]. Our compuational domain Γ_h approximating the smooth expected hypersurface Γ_0 will be a polyhedral surface

$$\Gamma_h=\bigcup_{T\in\mathcal{T}_h} T\subset U_\delta$$

consisting of finitely many non-degenerate triangles whose vertices are taken to lie on the surface Γ_0 and have the maximum diameter bounded above by $h>0$. The triangulation will be assumed to be shape regular and quasi-uniform, in the sense that the in ball radius ρ_K of each element K is uniformly bounded below by $\rho_K \geq ch$, for some constant $c>0$. In order to lift functions between the continuous and discrete surface, we shall assume that the projective mapping $a:\Gamma_h\to\Gamma_0$ decribed in (2.3.5) is bijective and define the lift and inverse lift of functions f and

g given over Γ_h and Γ_0 respectively by

$$f^l(a) = f(x(a)) \quad g^{-l}(x) = g(a(x)) \quad \text{for } a \in \Gamma_0, x \in \Gamma_h, \tag{2.6.1}$$

where $x(a)$ denotes the inverse of the projection mapping a. We introduce the linear finite element space on Γ_h

$$S_h = \{\phi_h \in C^0(\Gamma_h) \,|\, \phi_h|_T \in \mathbb{P}_1(T), T \in \mathcal{T}_h\} \tag{2.6.2}$$

and define the lifted finite element space by

$$S_h^l = \{\varphi_h \in C^0(\Gamma_0) \,|\, \varphi_h = \phi_h^l, \text{ for some } \phi_h \in S_h\}. \tag{2.6.3}$$

The finite element discretisation of the mean-weak formulation reads as follows.

Problem 2.6.1 (Semi-discrete scheme). *Find $U_h \in L^2(\Omega; S_h)$ such that*

$$\int_\Omega \int_{\Gamma_h} \mathcal{D}_{\Gamma_0}^{-l}(\omega)\nabla_{\Gamma_h} U_h(\omega) \cdot \nabla_{\Gamma_h}\phi_h(\omega) + U_h(\omega)\phi_h(\omega)\sqrt{g_{\Gamma_0}^{-l}(\omega)} = \int_\Omega \int_{\Gamma_h} f^{-l}(\omega)\phi_h(\omega)\sqrt{g_{\Gamma_0}^{-l}(\omega)} \tag{2.6.4}$$

for every $\phi_h \in L^2(\Omega; S_h)$.

In the context of the abstract framework, the finite dimensional space $\mathcal{V}_h$ is taken to be the finite element space S_h and the Hilbert spaces V_h, H_h and given by $H^1(\Gamma_h)$ and $L^2(\Gamma_h)$. Furthermore, the abstract sample-dependent discrete bilinear form $\tilde{a}_h(\omega; \cdot, \cdot) : H^1(\Gamma_h) \times H^1(\Gamma_h) \to \mathbb{R}$ and linear functional $\tilde{l}(\omega; \cdot) : L^2(\Gamma_h) \to \mathbb{R}$ are given by

$$\begin{aligned}
\tilde{a}_h(\omega; \chi_h, \phi_h) &= \int_{\Gamma_h} \mathcal{D}_{\Gamma_0}^{-l}(\omega)\nabla_{\Gamma_h}\chi_h \cdot \nabla_{\Gamma_h}\phi_h + \chi_h\phi_h\sqrt{g_{\Gamma_0}^{-l}(\omega)} \\
\tilde{l}_h(\omega; \chi_h) &= \int_{\Gamma_h} f^{-l}(\omega)\chi_h\sqrt{g_{\Gamma_0}^{-l}(\omega)}.
\end{aligned}$$

With the uniform bounds on the random coefficients (2.4.10), (2.4.11), we deduce that $\tilde{a}_h(\omega : \cdot, \cdot)$ is uniformly $L^2(\Omega; H^1(\Gamma_0))$-elliptic and bounded, and additionally $\tilde{l}(\omega; \cdot)$ is uniformly bounded as presumed in (2.5.2 - 2.5.4), and hence obtain existence and uniqueness of a semi-discrete solution to (2.6.4). We continue by checking the stated assumptions in the abstract error analysis. In particular, we begin with the norm equivalence (L1),(L2) of the lifting map $\Lambda_h : \mathcal{V}_h \to V$ given by $\Lambda_h\chi_h = \chi_h^l$. A proof of these estimates can be found in [42, Lemma 4.2].

Lemma 2.6.1 (Equivalence in norms of lifts). *There exists constants $c_1, c_2 > 0$ independent of h such that for any $\chi_h \in S_h$ with lift $\chi_h^l \in S_h^l$ we have*

$$\begin{aligned}
c_1\|\chi_h\|_{L^2(\Gamma_h)} \le \|\chi_h^l\|_{L^2(\Gamma_0)} \le c_2\|\chi_h\|_{L^2(\Gamma_h)}, \\
c_1\|\nabla_{\Gamma_h}\chi_h\|_{L^2(\Gamma_h)} \le \|\nabla_{\Gamma_0}\chi_h^l\|_{L^2(\Gamma_0)} \le c_2\|\nabla_{\Gamma_h}\chi_h\|_{L^2(\Gamma_h)}.
\end{aligned}$$

For the interpolation assumption (I1), we set the Hilbert space Z_0 consisting of functions of higher regularity to be $H^2(\Gamma_0)$. It follows from the Sobolev embedding that $H^2(\Gamma_0) \subset C^0(\Gamma_0)$

for $n \leq 3$ and therefore we can introduce the interpolation operator $I_h : H^2(\Gamma_0) \to S_h^l$ defined by

$$I_h \eta = \left(\hat{I}_h \eta^{-l}\right)^l \tag{2.6.5}$$

where $\hat{I}_h : C^0(\Gamma_h) \to S_h$ denotes the standard Lagrangian interpolatant defined element-wise on Γ_h. The following estimate was proved in [42, Lemma 4.3].

Lemma 2.6.2 (Interpolation estimate)**.** *Given any $\eta \in H^2(\Gamma_0)$, there exists a constant $c > 0$ independent of h such that*

$$\|\eta - I_h\eta\|_{L^2(\Gamma_0)} + h\|\nabla_{\Gamma_0}(\eta - I_h\eta)\|_{L^2(\Gamma_0)} \leq ch^2\|\eta\|_{H^2(\Gamma_0)}. \tag{2.6.6}$$

To derive the assumed bounds (P1),(P2) and (P3) on the approximation of the discrete bilinear forms, we first need a preliminary result on the order of approximation of the geometry, see [42, Lemma 4.1].

Lemma 2.6.3 (Geometric error bounds)**.** *Let $\delta_h^{\Gamma_0}$ denote the surface element corresponding to the transformation from Γ_0 to Γ_h under the lifting map $d\sigma(a(x)) = \delta_h(x)d\sigma_h(x)$ and define*

$$R_h^{\Gamma_0}(\omega) = \frac{1}{\delta_h^{\Gamma_0}}\left(\mathcal{D}_{\Gamma_0}^{-l}(\omega)\right)^{-1}\mathcal{P}_{\Gamma_0}(I - d^{\Gamma_0}\mathcal{H}^{\Gamma_0})\mathcal{P}_h\mathcal{D}_{\Gamma_0}^{-l}(\omega)\mathcal{P}_h(I - d^{\Gamma_0}\mathcal{H}^{\Gamma_0}), \tag{2.6.7}$$

where $\mathcal{P}_h := I - \nu_h \otimes \nu_h$ is the projection operator mapping onto the tangent space of the discrete surface Γ_h defined element-wise. Then we have the estimates

$$\|d^{\Gamma_0}\|_{L^\infty(\Gamma_h)} \leq ch^2 \tag{2.6.8}$$

$$\|1 - \delta_h^{\Gamma_0}\|_{L^\infty(\Gamma_h)} \leq ch^2 \tag{2.6.9}$$

$$\|(I - R_h^{\Gamma_0}(\omega))\mathcal{P}_{\Gamma_0}\|_{L^\infty(\Gamma_h)} \leq ch^2. \tag{2.6.10}$$

We can now bound the consistency error as follows.

Lemma 2.6.4 (Consistency error)**.** *Given any $(W_h, \phi_h) \in L^2(\Omega; S_h) \times L^2(\Omega; S_h)$ with lifts $(w_h, \varphi_h) \in L^2(\Omega; S_h^l) \times L^2(\Omega; S_h^l)$, we have*

$$|l(\varphi_h) - l_h(\phi_h)| \leq ch^2\|\varphi_h\|_{L^2(\Omega;L^2(\Gamma_0))} \tag{2.6.11}$$

$$|a(w_h, \varphi_h) - a_h(W_h, \phi_h)| \leq ch^2\|w_h\|_{L^2(\Omega;H^1(\Gamma_0))}\|\varphi_h\|_{L^2(\Omega;H^1(\Gamma_0))}. \tag{2.6.12}$$

Proof. Lifting the discrete integral in the linear functional $l_h(\cdot)$ onto the smooth surface Γ_0 with the projective mapping $a(\cdot)$ leads to

$$l(\varphi_h) - l_h(\phi_h) = \int_\Omega \int_{\Gamma_0} \left(1 - \frac{1}{\delta_h^{\Gamma_0}}\right) f(\omega)\varphi_h(\omega)\sqrt{g_{\Gamma_0}(\omega)}.$$

Hence with the uniform bound (2.4.11) on the random coefficient $\sqrt{g_{\Gamma_0}(\omega)}$ and the order h^2 approximation of the geometric pertubation (2.6.9), we obtain the estimate (2.6.11). For (2.6.12), we begin by applying the chain rule to lift $W_h(\omega, x) = w_h(\omega, a(x))$

$$\nabla_{\Gamma_h} W_h(\omega, x) = \mathcal{P}_h(x)(I - d^{\Gamma_0}(x)\mathcal{H}(x))\mathcal{P}_{\Gamma_0}(x)\nabla_{\Gamma_0} w_h(\omega, a(x)).$$

Suppressing the parameter x, we deduce

$$\begin{aligned}\mathcal{D}_{\Gamma_0}^{-l}(\omega)\nabla_{\Gamma_h} W_h(\omega) \cdot \nabla_{\Gamma_h}\phi_h(\omega) &= \mathcal{D}_{\Gamma_0}^{-l}(\omega)\mathcal{P}_h(I - d^{\Gamma_0}\mathcal{H})\mathcal{P}_{\Gamma_0}\nabla_{\Gamma_0} w_h(\omega, a) \cdot \mathcal{P}_h(I - d^{\Gamma_0}\mathcal{H})\mathcal{P}_{\Gamma_0}\nabla_{\Gamma_0}\varphi_h(\omega, a) \\ &= \mathcal{P}_{\Gamma_0}(I - d^{\Gamma_0}\mathcal{H})\mathcal{P}_h\mathcal{D}_{\Gamma_0}^{-l}(\omega)\mathcal{P}_h(I - d^{\Gamma_0}\mathcal{H})\mathcal{P}_{\Gamma_0}\nabla_{\Gamma_0} w_h(\omega, a) \cdot \nabla_{\Gamma_0}\varphi_h(\omega, a) \\ &= \delta_h^{\Gamma_0}\mathcal{D}_{\Gamma_0}^{-l}(\omega)R_h^{\Gamma_0}(\omega)\nabla_{\Gamma_0} w_h(\omega) \cdot \nabla_{\Gamma_0}\varphi_h(\omega).\end{aligned}$$

Therefore, we can express the pertubation error in the approximation of the bilinear form $a(\cdot, \cdot)$ by

$$\begin{aligned}a(w_h, \varphi_h) - a_h(W_h, \phi_h) &= \int_\Omega \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega)\left(\mathcal{P}_{\Gamma_0} - R_h^{\Gamma_0, l}(\omega)\right)\nabla_{\Gamma_0} w_h(\omega) \cdot \nabla_{\Gamma_0}\varphi_h(\omega) \\ &\quad + \int_\Omega \int_{\Gamma_0} \left(1 - \frac{1}{\delta_h^{\Gamma_0, l}}\right) w_h(\omega)\varphi(\omega)\sqrt{g_{\Gamma_0}(\omega)}\end{aligned}$$

and hence with the uniform bounds (2.4.10), (2.4.11) on the random coefficients and the geometric estimates (2.6.9), (2.6.10) we obtain (2.6.12). □

For the regularity assumption (R1) on the associated dual problem

$$a(\varphi, w(g)) = (g, \varphi)_{L^2(\Omega; L^2(\Gamma_0))} \quad \text{for all } \varphi \in L^2(\Omega; H^1(\Gamma_0)),$$

which due the symmetry of $\mathcal{D}_{\Gamma_0}$ and thus of $a(\cdot, \cdot)$ is precisely the mean-weak formulation, we have the results presented in Theorem 2.4.2.

2.6.2 The coupled elliptic system

We next apply the results from the abstract framework to our second model problem consisting of a coupled elliptic system

$$\begin{aligned}-\Delta u(\omega) + u(\omega) &= f(\omega) \quad \text{in } D(\omega) \\ \alpha u(\omega) - \beta v(\omega) + \frac{\partial u}{\partial \nu_\Gamma}(\omega) &= 0 \quad \text{on } \Gamma(\omega) \\ -\Delta_\Gamma v(\omega) + v(\omega) + \frac{\partial u}{\partial \nu_\Gamma}(\omega) &= f_\Gamma(\omega) \quad \text{on } \Gamma(\omega)\end{aligned}$$

posed over a random bulk-surface. Our proposed finite element discretisation of the reformulated system on the expected domain and the subsequent analysis will be based upon the approach taken and results derived in [34]. For the computational domain, we approximate the open bulk $D_0 \subset \mathbb{R}^{n+1}$ by a polyhedral

$$D_h = \bigcup_{K \in \mathcal{T}_h} K$$

consisting of closed $(n+1)-$simplices with maximum diameter uniformly bounded above by positive constant $h > 0$ and will assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. We denote the induced discrete surface $\Gamma_h = \partial D_h$ and the associated triangulation by

$$\Gamma_h = \bigcup_{T \in \mathcal{T}_h} T$$

and impose the same assumptions on $\mathcal{T}_h$ as were listed in the previous example. A piecewise diffeomorphic mapping $G_h : D_h \to D_0$ from the discrete bulk to the continuous can be constructed by fixing the interior simplices (simplices with at most one vertex on the boundary Γ_0) and using the projective mapping $a^{\Gamma_0}(\cdot)$ to define a diffeomorphism $\Lambda_{h,k} : K \to K^e$ between the boundary simplices K (simplices with at least two vertices on Γ_0) and the exact curved simplices K^e,

$$G_h|_K = \begin{cases} \Lambda_{h,K} & K \text{ boundary simplex} \\ id|_K & K \text{ interior simplex.} \end{cases} \tag{2.6.13}$$

Details on the precise form of $\Lambda_{h,K}$ can be found in [34]. We are therefore able to define lifts and inverse lifts of functions on the bulk domain by

$$\varphi_h^l(x) = \varphi_h(G_h^{-1}(x)) \quad x \in D_0 \tag{2.6.14}$$

$$\varphi^{-l}(x) = \varphi(G_h(x)) \quad x \in D_h. \tag{2.6.15}$$

Note that, the diffeomorphism $\Lambda_{h,K}$ is chosen such that the mapping G_h coincides with the projective mapping

$$G_h(x) = a^{\Gamma_0}(x) \quad x \in \partial D_h \tag{2.6.16}$$

on the boundary of the discrete bulk and hence the bulk lift agrees with the surface lifting map described in (2.6.1) on ∂D_h. For convenience, we will denote the sub-triangulation consisting of all boundary simplices by

$$\mathcal{B}_h = \{K \in \mathcal{T}_h \,|\, K \text{ is a boundary simplex}\}$$

and define the corresponding sets

$$B_h = \bigcup_{K \in \mathcal{B}_h} K \quad B_h^l = \bigcup_{K \in B_h} K^e \tag{2.6.17}$$

where the lifting maps G_h, G_h^{-1} differ from the identity mapping. We introduce the linear finite element spaces on the discrete bulk and discrete surface by

$$V_h = \{\phi_h \in C^0(D_h) \,|\, \phi_h|_K \in P^1(K) \text{ for all } K \in \mathcal{T}_h\} \tag{2.6.18}$$

$$S_h = \{\zeta_h \in C^0(\Gamma_h) \,|\, \zeta_h|_T \in P^1(T) \text{ for all } T \in \check{\mathcal{T}}_h\} \tag{2.6.19}$$

and denote the corresponding lifted finite element spaces by

$$V_h^l = \{\varphi_h = \phi_h^l \,|\, \phi_h \in V_h\} \quad S_h^l = \{\xi_h = \zeta_h^l \,|\, \zeta_h \in S_h\}. \tag{2.6.20}$$

An important feature of our finite element spaces is that the trace of a function $\phi_h \in V_h$ belongs to S_h and similarly the trace of $\varphi_h \in V_h^l$ belongs to S_h^l as a result of (2.6.16). The finite element discretisation of the mean-weak formulation then reads as follows.

Problem 2.6.2 (Semi-discrete problem). *Find a pair $(U_h, V_h) \in L^2(\Omega; V_h \times S_h)$ such that*

$$\begin{aligned}
&\alpha \int_\Omega \int_{D_h} \mathcal{D}^{-l}(\omega)\nabla U_h(\omega)\cdot\nabla\phi_h(\omega) + U_h(\omega)\phi_h(\omega)\sqrt{g^{-l}(\omega)} \\
&+\beta \int_\Omega \int_{\Gamma_h} \mathcal{D}_{\Gamma_0}^{-l}(\omega)\nabla_{\Gamma_h} V_h(\omega)\cdot\nabla_{\Gamma_h}\zeta_h(\omega) + V_h(\omega)\zeta_h(\omega)\sqrt{g_{\Gamma_0}^{-l}(\omega)} \\
&\int_\Omega \int_{\Gamma_h} (\alpha U_h(\omega) - \beta V_h(\omega))\,(\alpha\phi_h(\omega) - \beta\zeta_h(\omega))\,\sqrt{g_{\Gamma_0}^{-l}(\omega)} \\
&\qquad = \alpha \int_\Omega \int_{D_h} f^{-l}(\omega)\phi_h(\omega)\sqrt{g^{-l}(\omega)} + \beta \int_\Omega \int_{\Gamma_h} f_{\Gamma_0}^{-l}(\omega)\zeta_h(\omega)\sqrt{g_{\Gamma_0}^{-l}(\omega)}
\end{aligned}$$

for every $(\phi_h, \zeta_h) \in L^2(\Omega; V_h \times S_h)$.

Here the abstract finite dimensional space is $\mathcal{V}_h = V_h \times S_h$ and the Hilbert spaces V_h, H_h are given by $H^1(D_0) \times H^1(\Gamma_0)$ and $L^2(D_0) \times L^2(\Gamma_0)$ respectively. We denote the associated bilinear form and linear functional

$$a_h(\cdot,\cdot) : L^2(\Omega; V_h \times S_h) \times L^2(\Omega; V_h \times S_h) \to \mathbb{R} \quad l_h(\cdot) : L^2(\Omega; V_h \times S_h) \to \mathbb{R}$$

to be the respective left hand side and right hand side of the semi-discrete variational problem 2.6.2 By the uniform bounds on the random coefficients (2.4.23), (2.4.10), we deduce the existence and uniqueness of a semi-discrete solution using a similar argument to the continuous problem. We proceed in a similar manner and check that the assumptions of the abstract analysis are satisfied. The norm equivalence (L1), (L2) of the lifting mapping which in this setting $\Lambda_h : V_h \times S_h \to V_h^l \times S_h^l$ is given component-wise by

$$\Lambda_h\left((\phi_h, \zeta_h)\right) = (\phi_h^l, \zeta_h^l), \tag{2.6.21}$$

follows from the estimates on the surface lifting map given Lemma 2.6.1 in combination with the following bulk lifting norm equivalence derived in [34, Proposition 4.9]

Lemma 2.6.5 (Bulk lift estimates). *There exists constants $c_1, c_2 > 0$ independent of h, such that for any $\phi_h : D_h \to \mathbb{R}$ with lift $\varphi_h = \phi_h^l : D_0 \to \mathbb{R}$ we have*

$$\begin{aligned}
c_1\|\phi_h\|_{L^2(D_h)} \le \|\varphi_h\|_{L^2(D_0)} \le c_2\|\phi_h\|_{L^2(D_h)} \\
c_1\|\phi_h\|_{H^1(D_h)} \le \|\varphi_h\|_{H^1(D_0)} \le c_2\|\phi_h\|_{H^1(D_h)}.
\end{aligned}$$

For the interpolation assumption (I1), we set the abstract function space $Z_0 = H^2(D_0) \times H^2(\Gamma_0)$ and define the interpolation operator component-wise

$$I_h(\eta, \xi) = \Big((\tilde{I}_h \eta^{-l})^l, (\tilde{I}_h \xi^{-l})^l \Big) \tag{2.6.22}$$

with $\tilde{I}_h$ denoting the standard Lagrangian intepolation operator and have the following estimate [34, Proposition 5.4].

Lemma 2.6.6 (Interpolation estimate)**.** *There exists a well-defined interpolation operator*

$$I_h : H^2(D_0) \times H^2(\Gamma_0) \to V_h^l \times S_h^l$$

such that for any $(\eta, \xi) \in H^2(D_0) \times H^2(\Gamma_0)$ *we have*

$$\|(\eta, \xi) - I_h(\eta, \xi)\|_{L^2(D_0)\times L^2(\Gamma_0)} + h\|(\eta, \xi) - I_h(\eta, \xi)\|_{H^1(D_0)\times H^1(\Gamma_0)} \leq ch^2 \|(\eta, \xi)\|_{H^2(D_0)\times H^2(\Gamma_0)}. \tag{2.6.23}$$

The next step will entail bounding the consistency error arising from the geometric approximation of the domain. Estimates for the surface pertubation have previously been given in Lemma 2.6.3. For the bulk approximation, we recall that the lifting mapping $G_h : D_h \to D_0$ is defined to be the identity on interior simplices and a C^1-diffeomorphism for simplices near the boundary. Therefore the corresponding bulk error will be comprised of two parts; the first part will be related to the smallness of the neighbourhood around Γ_0 in which the lifted boundary simplices lie in and the second part is the associated geometric error of the boundary simplices approximating the corresponding exact curved simplex. We begin with the latter and state geometric bulk estimates on the diffeomorphic mapping G_h, for which a proof of the bounds (2.6.24) and (2.6.25) can be found in [34, Proposition 4.7].

Lemma 2.6.7 (Geometric bulk estimates)**.** *Let* $\delta_h^{D_0} = |det(\nabla G_h)|$ *be the volume element corresponding to the transformation* $G_h : D_h \to D_0$ *and set*

$$R_h^{D_0}(\omega) = \frac{1}{\delta_h^{D_0}} \Big(\mathcal{D}^{-l}(\omega)\Big)^{-1} \nabla G_h \mathcal{D}^{-l}(\omega) \nabla G_h^\top.$$

Then we have the following estimates for a constant $c > 0$ *independent of* ω*,*

$$\|\nabla G_h - I\|_{L^\infty(D_h)} \leq ch \tag{2.6.24}$$

$$\|\delta_h^{D_0} - 1\|_{L^\infty(D_h)} \leq ch \tag{2.6.25}$$

$$\|R_h^{D_0}(\omega) - I\|_{L^\infty(D_h)} \leq ch. \tag{2.6.26}$$

Proof. The estimate (2.6.26) follows from the observation

$$R_h^{D_0}(\omega) - I = \frac{1}{\delta_h^{D_0}} \left(\mathcal{D}^{-l}(\omega)\right)^{-1} \nabla G_h \mathcal{D}^{-l}(\omega) \left(\nabla G_h^\top - I\right) + \frac{1}{\delta_h^{D_0}} \left(\mathcal{D}^{-l}(\omega)\right)^{-1} \left(\nabla G_h - I\right) \mathcal{D}^{-l}(\omega)$$
$$+ \left(\frac{1}{\delta_h^{D_0}} - 1\right) I.$$

and the uniform bounds (2.4.23) on the random coefficient $\mathcal{D}(\omega)$. ☐

To obtain a bound on the open neighbourhood containing the boundary simplices, we have the subsequent narrow band inequality [34, Lemma 4.10].

Lemma 2.6.8 (Narrow band trace inequality)**.** *Given any $\delta < \delta_{\Gamma_0}$, let $\mathcal{N}_\delta$ be a narrow band in the interior domain D_0 around the boundary Γ_0 defined by*

$$\mathcal{N}_\delta = \{x \in D_0 \,|\, -\delta < d(x) < 0\}. \tag{2.6.27}$$

Then for any $\eta \in H^1(D_0)$ we have

$$\|\eta\|_{L^2(\mathcal{N}_\delta)} \leq c\delta^{\frac{1}{2}} \|\eta\|_{H^1(D_0)}.$$

The consistency error can now be bounded as follows, where we have developed upon the results presented in [34], to our given stochastic setting.

Lemma 2.6.9 (Consistency error)**.** *Assume $f \in L^2(\Omega; H^1(D_0))$. Then for any $\phi_h, W_h \in L^2(\Omega; V_h)$ and $\zeta_h, X_h \in L^2(\Omega; S_h)$ with corresponding lifts φ_h, w_h and ξ_h, χ_h we have*

$$|l(\varphi_h, \xi_h) - l_h(\phi_h, \zeta_h)| \leq ch^2 \|(f, f_{\Gamma_0})\|_{L^2(\Omega; H^1(D_0) \times L^2(\Gamma_0))} \|(\varphi_h, \xi_h)\|_{L^2(\Omega; H^1(D_0) \times H^1(\Gamma_0))} \tag{2.6.28}$$

$$\begin{aligned} &|a\left((\varphi_h, \xi_h), (w_h, \chi_h)\right) - a_h((\phi_h, \zeta_h), (W_h, X_h))| \\ &\qquad \leq ch \|(\varphi_h, \zeta_h)\|_{L^2(\Omega; H^1(D_0) \times H^1(\Gamma_0))} \|(w_h, \chi_h)\|_{L^2(\Omega; H^1(D_0) \times H^1(\Gamma_0))}. \end{aligned} \tag{2.6.29}$$

Furthermore, for any $\varphi, w \in L^2(\Omega; H^2(D_0))$ and $\xi, \chi \in L^2(\Omega; H^2(\Gamma_0))$ with inverse lifts φ^{-l}, w^{-l} and ξ^{-l}, χ^{-l} we have

$$\begin{aligned} &|a\left((\varphi, \xi), (w, \chi)\right) - a_h \left((\varphi^{-l}, \xi^{-l}), (w^{-l}, \chi^{-l})\right)| \\ &\qquad \leq ch^2 \|(\varphi, \xi)\|_{L^2(\Omega; H^2(D_0) \times H^2(\Gamma_0))} \|(w, \chi)\|_{L^2(\Omega; H^2(D_0) \times H^2(\Gamma_0))}. \end{aligned} \tag{2.6.30}$$

Proof. For the estimate (2.6.28), we begin by lifting the discrete integrals in $l_h(\cdot)$ onto their respective continuous counterparts recalling that the set of all boundary simplices B_h is the region in which the diffeomorphic mapping G_h differs from the identity and thus where $\delta_h^{D_0} =$

$det(\nabla G_h) \neq 1$,

$$
\begin{aligned}
&l(\varphi_h, \xi_h) - l_h(\phi_h, \zeta_h) \\
&\qquad = \alpha \int_\Omega \int_{D_0} \left(1 - \frac{1}{\delta_h^{D_0,l}}\right) f(\omega)\varphi_h(\omega)\sqrt{g(\omega)} + \beta \int_\Omega \int_{\Gamma_0} \left(1 - \frac{1}{\delta_h^{\Gamma_0,l}}\right) f_{\Gamma_0}(\omega)\xi_h(\omega)\sqrt{g_{\Gamma_0}(\omega)} \\
&\qquad = \alpha \int_\Omega \int_{B_h^l} \left(1 - \frac{1}{\delta_h^{D_0,l}}\right) f(\omega)\varphi_h(\omega)\sqrt{g(\omega)} + \beta \int_\Omega \int_{\Gamma_0} \left(1 - \frac{1}{\delta_h^{\Gamma_0,l}}\right) f_{\Gamma_0}(\omega)\xi_h(\omega)\sqrt{g_{\Gamma_0}(\omega)}.
\end{aligned}
$$

Substituting the geometric bulk and surface estimates (2.6.25), (2.6.9) with the uniform bounds on the random coefficients (2.4.23), (2.4.10) leads to

$$
|l(\varphi_h, \xi_h) - l_h(\phi_h, \zeta_h)| \lesssim h\|f\|_{L^2(\Omega;L^2(B_h^l))}\|\varphi_h\|_{L^2(\Omega;L^2(B_h^l))} + h^2\|f_{\Gamma_0}\|_{L^2(\Omega;L^2(\Gamma_0))}\|\xi_h\|_{L^2(\Omega;L^2(\Gamma_0))}.
$$

To obtain a bound of order h^2 on the bulk term, we will now apply the narrow trace band inequality. We choose $\delta > 0$ such that $0 < h < \delta < ch$ for some constant $c > 0$, thus giving

$$
\|f\|_{L^2(\Omega;L^2(B_h^l))} \leq \|f\|_{L^2(\Omega;L^2(\mathcal{N}_\delta))} \leq c\delta^{\frac{1}{2}}\|f\|_{L^2(\Omega;H^1(D_0))} \leq ch^{\frac{1}{2}}\|f\|_{L^2(\Omega;H^1(D_0))}. \tag{2.6.31}
$$

With a similar estimate on the test function φ_h, we obtain (2.6.28). For (2.6.29) and (2.6.30), we apply the chain rule to the lifts $\varphi_h(\omega, G_h(x)) = \phi_h(\omega, x)$ and $w_h(\omega, G_h(x)) = W_h(\omega, x)$ to deduce

$$
\begin{aligned}
\mathcal{D}^{-l}(\omega, x)\nabla\phi_h(\omega, x) \cdot \nabla W_h(\omega, x) &= \mathcal{D}^{-1}(\omega, x)\nabla G_h^\top(x)\nabla\varphi_h(\omega, G_h(x)) \cdot \nabla G_h^\top(x)\nabla w_h(\omega, G_h(x)) \\
&= \nabla G_h(x)\mathcal{D}^{-l}(\omega, x)\nabla G_h^\top(x)\nabla\varphi_h(\omega, G_h(x)) \cdot \nabla w_h(\omega, G_h(x)) \\
&= \delta_h^{D_0}(x)\mathcal{D}^{-l}(\omega, x)R_h^{D_0}(\omega, x)\nabla\varphi_h(\omega, G_h(x)) \cdot \nabla w_h(\omega, G_h(x)).
\end{aligned}
$$

We can therefore express the perturbation error in our approximation of $a(\cdot, \cdot)$ as follows

$$
\begin{aligned}
&a((\varphi_h,\xi_h), (w_h, \chi_h)) - a_h((\phi_h, \zeta_h), (W_h, X_h)) \\
&\quad = \alpha \int_\Omega \int_{B_h^l} \mathcal{D}(\omega)\left(I - R_h^{D_0,l}(\omega)\right)\nabla\varphi_h(\omega) \cdot \nabla w_h(\omega) + \left(1 - \frac{1}{\delta_h^{D_0,l}}\right)\varphi_h(\omega)w_h(\omega)\sqrt{g(\omega)} \\
&\quad + \beta \int_\Omega \int_{\Gamma_0} \mathcal{D}_{\Gamma_0}(\omega)\left(\mathcal{P}_{\Gamma_0} - R_h^{\Gamma_0,l}\right)\nabla_{\Gamma_0}\xi_h(\omega) \cdot \nabla_{\Gamma_0}\chi_h(\omega) + \left(1 - \frac{1}{\delta_h^{\Gamma_0,l}}\right)\xi_h(\omega)\chi_h(\omega)\sqrt{g_{\Gamma_0}(\omega)} \\
&\quad + \int_\Omega \int_{\Gamma_0} \left(1 - \frac{1}{\delta_h^{\Gamma_0,l}}\right)(\alpha\varphi_h(\omega) - \beta\xi_h(\omega))(\alpha w_h(\omega) - \beta\chi(\omega))\sqrt{g_{\Gamma_0}(\omega)}.
\end{aligned}
$$

Here we have again used the fact that the diffeomorphic mapping G_h is the identity on interior simplices and consequently $\delta_h^{D_0} = 1$ and $R_h^{D_0} = I$ on $D_h \setminus B_h$. We now apply the geometric estimates and bounds on the random coefficients to obtain

$$
\begin{aligned}
|a((\varphi_h,\xi_h),(w_h,\chi_h)) - a_h((\phi_h,\zeta_h),(W_h,X_h))| \lesssim\; & h\|\varphi_h\|_{L^2(\Omega;H^1(B_h^l))}\|w_h\|_{L^2(\Omega;H^1(B_h^l))} \\
& + h^2\|\xi_h\|_{L^2(\Omega;H^1(\Gamma_0))}\|\chi_h\|_{L^2(\Omega;H^1(\Gamma_0))} \\
& + h^2\|\alpha\varphi_h - \beta\xi_h\|_{L^2(\Omega;L^2(\Gamma_0))}\|\alpha w_h - \beta\chi_h\|_{L^2(\Omega;L^2(\Gamma_0))}
\end{aligned}
$$

For the last term, we observe by the boundedness of the trace operator $\|f\|_{L^2(\Gamma_0)} \leq c_T\|f\|_{H^1(D_0)}$ that

$$
\begin{aligned}
&\|\alpha\varphi_h - \beta\xi_h\|_{L^2(\Omega;L^2(\Gamma_0))}\|\alpha w_h - \beta\chi_h\|_{L^2(\Omega;L^2(\Gamma_0))} \\
&\quad \leq \left(\alpha c_T\|\varphi_h\|_{L^2(\Omega;H^1(D_0))} + \beta\|\xi_h\|_{L^2(\Omega;L^2(\Gamma_0))}\right)\left(\alpha c_T\|w_h\|_{L^2(\Omega;H^1(D_0))} + \beta\|\chi_h\|_{L^2(\Omega;L^2(\Gamma_0))}\right) \\
&\quad \lesssim \|(\varphi_h,\xi_h)\|_{L^2(\Omega;H^1(D_0)\times L^2(\Gamma_0))}\|(w_h,\chi_h)\|_{L^2(\Omega;H^1(D_0)\times L^2(\Gamma_0))}.
\end{aligned}
$$

Examining the bulk term, we see that we are unable to apply the narrow band inequality to the derivative of $\varphi_h(\omega)$ and $w_h(\omega)$ since the functions only belong to the space $V_h \subset H^1(D_0)$, resulting the bound of order h given in (2.6.29). However, considering sufficiently regular functions $\varphi, w \in L^2(\Omega; H^2(D_0))$, we are able to employ the result attaining an estimate of order h^2 given in (2.6.30). □

The regularity assumption (R1) on the associated dual problem follows again from the symmetry of the bilinear for $a(\cdot,\cdot)$ and the previously derived regularity result given in Theorem 2.4.4. Hence all the assumptions of the abstract theory are satisfied and we have the stated convergence rate given in Theorem 2.5.3.

2.7 Numerical results

In this section, we numerically verify the stated convergence rates of the two proposed finite element discretisations of the reformulated model elliptic problems. In both cases, the numerical scheme has been implemented in DUNE [4, 27].

2.7.1 Random Surface

As a model for the random surface $\Gamma(\omega)$, we consider graphical representation over the unit sphere $\Gamma_0 = S^2$

$$
\Gamma(\omega) = \{x + h(\omega,x)\nu^{\Gamma_0}(x) \,|\, x \in \Gamma_0\}, \tag{2.7.1}
$$

where the prescribed height function $h(\omega,\cdot) : \Gamma_0 \to \mathbb{R}$, will take the form of a truncated spherical harmonic expansion

$$
h(\omega,x) = \epsilon_{tol}\sum_{m<6}\sum_{|l|\leq m}\lambda_{l,m}(\omega)Y_l^m(\theta,\phi) \quad x = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{2.7.2}
$$

with independent, uniformly distributed random coefficients $\lambda_{l,m} \sim U(-1,1)$. Here $\epsilon_{tol} > 0$ is a parameter controlling the maximum deviation of the fluctuating surface which in practice will be set to $\epsilon_{tol} = 0.1$ and Y_l^m denotes the spherical harmonic function of degree l and order m, which correspond to the eigenvalues of the Laplace-Beltrami operator. For further details on exact form of the spherical harmonics, we refer the reader to [3, 45]. Realisations of the random surface for different samples are given below in Figure 2.4. To numerical verify the convergence

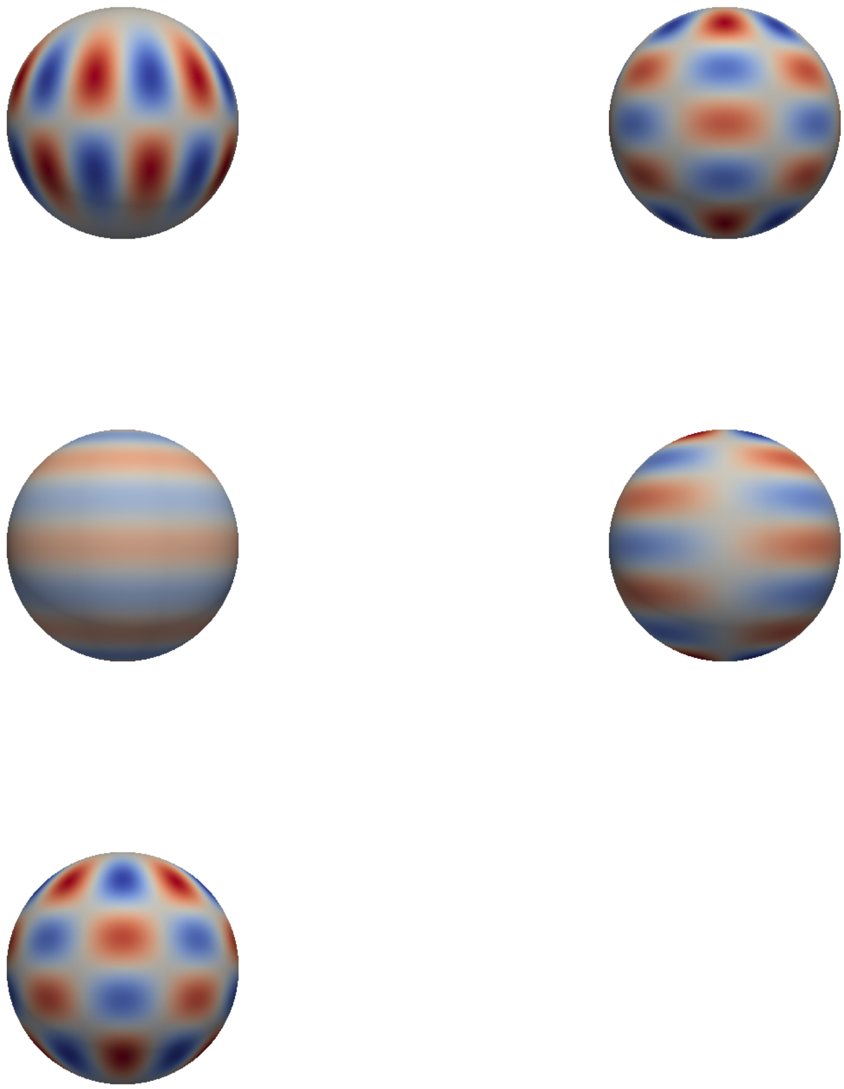

Figure 2.3: Plots of some the spherical harmonics of order 8 considered.

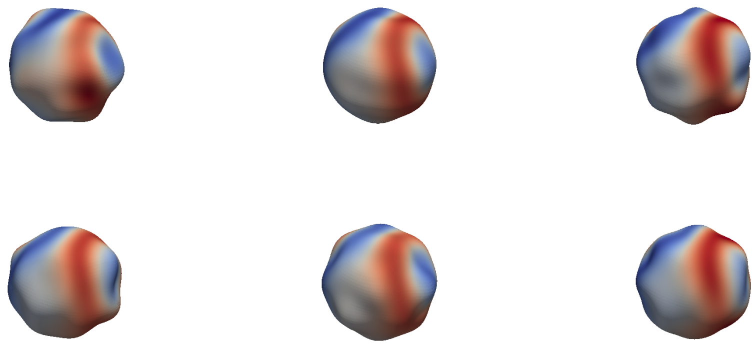

Figure 2.4: Realisations of the path-wise solution on the associated realisation of the random surface.

rate, we set the exact pull-back solution to be given by

$$\hat{u}(\omega, x) = sin(\pi(x^2 - 1)y(z - 1)) + \sigma_{tol}\nu_1(\omega)cos(\pi z(y + 1)) + \sigma_{tol}\nu_2(\omega)sin(\pi(x + y)z^2)$$

with $\nu_1, \nu_2 \sim U(-1, 1)$ and $\sigma_{tol} > 0$ a constant controlling the largest deviation of pathwise solution. This in turn determines the random data $\hat{f}$ given in the reformulated elliptic equation (2.4.4). We observe the following errors for the approximation $\mathbb{E}[\hat{u}] - E_M[\hat{u}_h]$ in $L^2(\Omega^M; L^2(\Gamma_0))$ and $L^2(\Omega^M; H^1(\Gamma_0))$ and thus the stated convergence results.

h	M	$E_{L^2(\Gamma_0)}$	$eoc(h)$	$eoc(M)$
0.171499	1	0.776832	-	-
0.0877058	16	0.387486	1.03722	-0.250864
0.0441081	256	0.106022	1.88556	-0.467444
0.0220863	4096	0.0267303	1.99202	-0.496955

Table 2.1: Error in $L^2(\Omega^M; L^2(\Gamma_0))$.

h	M	$E_{H^1(\Gamma_0)}$	$eoc(h)$	$eoc(M)$
0.171499	64	4.89172	-	-
0.0877058	256	3.68809	0.421176	-0.203734
0.0441081	1024	1.90402	0.961875	-0.476911
0.0220863	4096	0.961782	0.987348	-0.492633

Table 2.2: Error in $L^2(\Omega^M; H^1(\Gamma_0))$.

2.7.2 Random bulk-surface

For the coupled-elliptic system on a random bulk-surface, we adopt a similar approach to the random surface numerical example and prescribe the curved boundary to the random bulk $D(\omega)$ which for simplicity is taken to lie in $\mathbb{R}^2$, as a graph

$$\Gamma(\omega) = \{x + h(\omega, x)\nu^{\Gamma_0}(x) \,|\, x \in S^1\} \tag{2.7.3}$$

over the unit circle. Here the random height function will given by a truncated Fourier series

$$h(\omega, x) = \sum_{n=1}^{6} \lambda_n(\omega) cos(n\theta) + \hat{\lambda}_n(\omega) sin(n\theta) \quad x = (cos(\theta), sin(\theta)) \in S^1,$$

with independent, uniformly distributed random coefficients $\lambda_n, \hat{\lambda}_n \sim U(-1, 1)$. We extend the given boundary process in the normal direction into the interior with a sufficiently smooth blending function to form the stochastic domain mapping

$$\phi(x, \omega) = x + L_\delta(|x - a^{\Gamma_0}(x)|)h(a^{\Gamma_0}(x), \omega)\nu^{\Gamma_0}(a^{\Gamma_0}(x)) \quad x \in \overline{B_1(0)}. \tag{2.7.4}$$

Here the precise form of the chosen blending function $L_\delta(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by

$$L_\delta(x) = \begin{cases} exp\left(\frac{-x^2}{\delta^2 - x^2}\right) & \text{if } x < \delta, \\ 0 & \text{if } x \geq \delta. \end{cases}$$

Realisations of the image of the reference domain mappped under the stochastic mapping (2.7.4) are provided in Figure 2.5.

Figure 2.5: Realisations on pathwise solution on the random bulk-surface.

We set the pull-back of the path-wise bulk solution to be given by

$$\hat{u}(\omega, x) = sin(\pi xy)cos(\pi y^2) + \epsilon_{tol}\lambda(\omega)cos(\pi xy)$$

with uniformly distributed random coefficient $\lambda \sim U(-1, 1)$ and $\epsilon_{tol} = 0.1$. This determines the pull-back of the path-wise surface solution $\hat{v}$ by the reformulated Robin boundary condition

$$\alpha\hat{u}(\omega) - \beta\hat{v}(\omega) + \frac{\sqrt{g(\omega)}}{\sqrt{g_{\Gamma_0}(\omega)}} G^{-1}(\omega)\nu^{\Gamma_0} \cdot \nabla\hat{u}(\omega) = 0 \quad \text{on } \Gamma_0,$$

from which the data f and $\hat{f}_{\Gamma_0}$ can then be computed. Note that in practice, the expectation $\mathbb{E}[\hat{v}]$ and its surface derivative are approximated with Monte-Carlo sampling to sufficiently high accuracy. We observe the following errors and experimental order of convergence for the approximations of the bulk $\mathbb{E}[\hat{u}] - E_M[\hat{u}_h]$ and the surface $\mathbb{E}[\hat{v}] - E_M[\hat{v}_h]$ mean solutions.

h	M	Bulk $E_{L^2(D_0)}$	$eoc(h)$	$eoc((M)$	Surface $E_{L^2(\Gamma_0)}$	$eoc(h)$	$eoc((M)$
0.27735	1	0.619144	-	-	5.0787	-	-
0.156174	16	0.198298	1.98249	-0.410651	1.06707	2.71654	-0.562702
0.0830455	256	0.0540441	2.05828	-0.468866	0.28356	2.0983	-0.477981
0.0428353	4096	0.0152612	1.91003	-0.456067	0.0723061	2.06414	-0.492866

Table 2.3: Errors in $L^2(\Omega^M; L^2(D_0))$ and $L^2(\Omega^M; L^2(\Gamma_0))$.

h	M	Bulk $E_{L^2(D_0)}$	$eoc(h)$	$eoc(M)$	Surface $E_{L^2(\Gamma_0)}$	$eoc(h)$	$eoc(M)$
0.27735	64	3.41133	-	-	15.5792	-	-
0.156174	256	2.17523	0.783494	-0.324584	7.85391	1.1926	-0.494068
0.0830455	1024	1.08874	1.09584	-0.499252	4.20041	0.990894	-0.451441
0.0428353	4096	0.55599	1.01511	-0.484767	2.12783	1.02727	-0.490574

Table 2.4: Errors in $L^2(\Omega^M; H^1(D_0))$ and $L^2(\Omega^M; H^1(\Gamma_0))$.

Chapter 3

An extension of the domain mapping method to advection-diffusion equations posed over randomly evolving curved domains

This chapter will focus on the extending the domain mapping method to advection-diffusion equations posed over randomly evolving surfaces and randomly evolving bulk-surface systems. We will begin by describing the geometry of the randomly evolving surface and derive an advection-diffusion equation on the randomly evolving surface via a conservation law. We will then proceed by deriving expressions for the pull-back of time dependent quantities, such as the material derivative, defined over the randomly evolving curved domain, onto a deterministic reference domain via the prescribed domain mapping. With these computations, we will then present two model problems consisting of an advection-diffusion equation posed over a randomly evolving surface and a coupled advection-diffusion system posed over a randomly evolving bulk-surface, and in each case we will apply the domain mapping method to reformulate the respective problems onto a deterministic reference domain. Following on from this, we will present an abstract analysis of the general form of the reformulated equations which arise after transforming the original advection-diffusion problem on a random curved domain onto a deterministic reference domain. A finite element discretisation coupled with the single-level Monte Carlo method, will subsequently be presented in the abstract setting, and optimal error estimates will be derived. We will then conclude by numerically verifying the stated convergence rates with our two model problems.

3.1 Advection-diffusion equations on randomly evolving curved domains and the domain mapping method

In this section, we describe the geometry of the randomly evolving surface based upon a random level-set representation. We further derive standard conservation laws on the random surface as well as the random bulk-surface, developing upon the derivations presented in [42, Section 5.2] for the deterministic analogues. This will lead to the consideration of two prototypical advection-diffusion equations, respectively posed on a randomly evolving surface and a randomly evolving coupled system, that will serve as model examples for the extended domain mapping method. Note that throughout this chapter, we will let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete separable probability space. For further details on the formulation and analysis of advection-diffusion equations on an evolving deterministic surface, including a more in-depth discussion on the description of the evolving geometry, we refer the reader to [42].

3.1.1 Randomly evolving surface

For a.e. $\omega \in \Omega$ and each $t \in [0,T]$, where $T > 0$ is a fixed constant, let $\Gamma(\omega;t) = \Gamma_\omega(t)$ be a compact, orientable C^2-hypersurface in $\mathbb{R}^{n+1}$ and let $\Gamma_{\omega,0} = \Gamma_\omega(0)$ denote the initial surface, which may also be treated as random. It follows that we may represent $\Gamma_\omega(t)$ as the zero level-set

$$\Gamma_\omega(t) = \{x \in \mathcal{N}_\omega(t) \,|\, d(\omega;x,t) = 0\}$$

of a random field $d(\omega;x,t)$ with $x \in \mathbb{R}^{n+1}, t \in [0,T]$, where $\mathcal{N}_\omega(t) \subset \mathbb{R}^{n+1}$ is an open set in which $\nabla d(\omega;x,t) \neq 0$. We assume that realisations of the random field $d(\omega;\cdot,\cdot) = d_\omega(\cdot,\cdot)$ are sufficiently regular in the associated realisation of the random space-time domain

$$\mathcal{N}_{\omega,T} = \bigcup_{t\in[0,T]} \mathcal{N}_\omega(t) \times \{t\},$$

to ensure that the evolution of realisations of the random surface as well as the surface $\Gamma_\omega(t)$ at each $t \in [0,T]$, are both sufficiently smooth. More precisely, we assume for *a.e.* ω,

$$d_\omega,\, \partial_{x_i} d_\omega,\, \partial_{x_i}\partial_{x_j} d_\omega,\, \partial_t d_\omega \in C^0\left(\mathcal{N}_{\omega,T}\right) \quad i,j = 1,...,n+1. \tag{3.1.1}$$

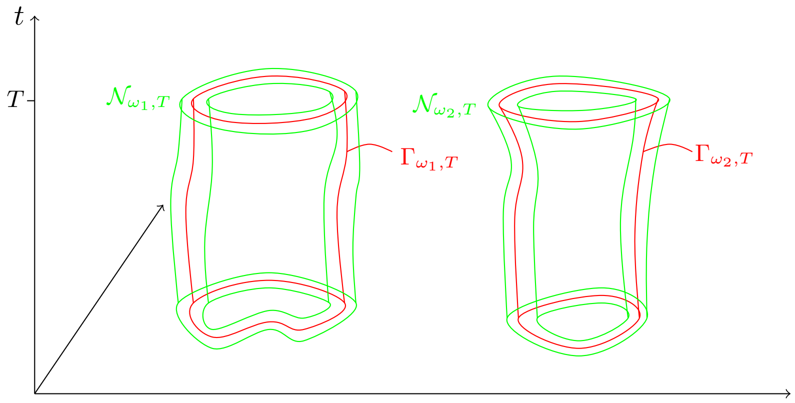

Figure 3.1: Realisations of the evolution of the random surface Γ_ω from the initial random domain $\Gamma_{\omega,0}$ and the associated space-time neighbourhood $\mathcal{N}_{\omega,T}$.

Since $\Gamma_\omega(t)$ is compact and thus encloses an open bulk domain, we will furthermore assume that $d(\omega; x, t) < 0$ in the interior and will fix the orientation of each $\Gamma_\omega(t)$ to be in the outer direction by choosing the random unit normal vector field as

$$\nu^{\Gamma_\omega}(\omega; x, t) = \frac{\nabla d(\omega; x, t)}{|\nabla d(\omega; x, t)|}.$$

The random normal velocity V^{Γ_ω} of Γ_ω, which is sufficient in describing the evolution of the random surface, is subsequently given by

$$V^{\Gamma_\omega}(\omega; x, t) = -\frac{d_t(\omega; x, t)}{|\nabla d(\omega; x, t)|}$$

and we have $V^{\Gamma_\omega}(\omega; \cdot, t) \in C^1(\Gamma_\omega(t))$ for $a.e.\, \omega$ and all $t \in [0, T]$, due the smoothness assumption (3.1.1) on $d(\omega; \cdot, \cdot)$. We denote the corresponding random normal velocity vector field by

$$v_{\nu^{\Gamma_\omega}}(\omega; x, t) = V^{\Gamma_\omega}(\omega; x, t)\nu^{\Gamma_\omega}(\omega; x, t).$$

A real-valued random field f may now be defined on the randomly evolving surface Γ, as the restriction of a random field $\bar{f} : \Omega \times X \times [0, T] \to \mathbb{R}$ onto realisations of the random space-time surface

$$\Gamma_{\omega,T} = \bigcup_{t \in [0,T]} \Gamma_\omega(t) \times \{t\}, \tag{3.1.2}$$

where $X \subset \mathbb{R}^{n+1}$ is an auxiliary set such that $\Gamma_{\omega,T} \subset X \times [0, T]$ for a.e. ω. It is worthwhile noting that the purpose of the above extension is only to give a precise definition of a random field on a random domain. In particular, all of the subsequent definitions may be shown to be independent of the extension provided. We define the normal time derivative of a scalar random field on Γ pathwise by

$$(\partial^\circ f)(\omega; \cdot, t) = (\partial^\circ f_\omega)(\cdot, t) = \partial_t f_\omega(\cdot, t) + \nabla f_\omega(\cdot, t) \cdot v_{\nu^{\Gamma_\omega}}(\cdot, t),$$

and for convenience, denote realisations of a given random field by $f_\omega(\cdot,t) = f(\omega;\cdot,t)$. We next introduce a notion of a random tangential velocity field v_τ over the randomly evolving surface Γ, as a vector-valued random field with realisations defined over the space-time surface $v_{\tau,\omega} : \Gamma_{\omega,T} \to \mathbb{R}^{n+1}$ which at each $t \in [0,T]$ maps into the following tangent space

$$v_{\tau,\omega}(\cdot,t) : \Gamma_\omega(t) \to T\Gamma_\omega(t).$$

Equivalently, a random tangential velocity field may be defined as the restriction onto the randomly evolving surface Γ, of the projection of a general random velocity field $\tilde{v} : \Omega \times X \times [0,T] \to \mathbb{R}^{n+1}$ onto the tangent space, i.e. $v_{\tau,\omega}(\cdot,t) = \mathcal{P}_{\Gamma_\omega(t)}\tilde{v}_\omega(\cdot,t)$. Considering a general random velocity field $v_\omega = v_{\nu^{\Gamma_\omega}} + v_{\tau,\omega}$ with a tangential component, we may similarly define the material derivative associated to v of a scalar random field f on Γ, pathwise via

$$\left(\partial_v^\bullet f\right)(\omega;\cdot,t) = \partial_{v_\omega}^\bullet f_\omega(\cdot,t) = \partial_t f_\omega(\cdot,t) + \nabla f_\omega(\cdot,t)\cdot v_{\tau,\omega}(\cdot,t)$$

and observe that $\partial_v^\bullet f_\omega = \partial^\circ f_\omega + \nabla_{\Gamma_\omega} f_\omega \cdot v_{\tau,\omega}$.

3.1.2 A conservation law on the randomly evolving surface

Let u denote a scalar random field, defined for each $\omega \in \Omega$ over the associated realisation of the random space-time surface $u_\omega : \Gamma_{\omega,T} \to \mathbb{R}$ described in (3.1.2), that represents the random density of some quantity which exists on the randomly evolving surface, and let q denote a random tangential surface flux over Γ. The conservation law we wish to consider states that for every portion $\mathcal{M}_\omega(t)$ of $\Gamma_\omega(t)$ evolving under the random normal velocity field $v_{\nu^{\Gamma_\omega}}$, we have

$$\frac{d}{dt}\int_{\mathcal{M}_\omega(t)} u_\omega = -\int_{\partial\mathcal{M}_\omega(t)} q_\omega \cdot \mu_\omega,$$

where $\mu_\omega(t)$ denotes the co-normal vector to $\mathcal{M}_\omega(t)$, that is the outer unit normal vector to $\mathcal{M}_\omega(t)$ which is tangent to the hypersurface $\Gamma_\omega(t)$, see Figure 3.2 for an illustration. As a consequence of the transport property (A.2.2) and the integration by parts formula for surfaces (A.2.1), recalling that the flux q_ω is assumed to be tangential to the surface, we have

$$\int_{\mathcal{M}_\omega(t)} \partial^\circ u_\omega + u_\omega \nabla_{\Gamma_\omega}\cdot v_{\nu^{\Gamma_\omega}} = -\int_{\mathcal{M}_\omega(t)} \nabla_{\Gamma_\omega}\cdot q_\omega - \int_{\mathcal{M}_\omega(t)} q_\omega \cdot H^{\Gamma_\omega}\nu^{\Gamma_\omega} = -\int_{\mathcal{M}_\omega(t)} \nabla_{\Gamma_\omega}\cdot q_\omega,$$

and thus obtain the following pointwise conservation law

$$\partial^\circ u_\omega + u_\omega \nabla_{\Gamma_\omega}\cdot v_{\nu^{\Gamma_\omega}} + \nabla_{\Gamma_\omega}\cdot q_\omega = 0 \tag{3.1.3}$$

on realisations of the randomly evolving surface Γ_ω. By considering a random diffusive flux q^{dif} defined *a.e.* by $q_\omega^{dif} = -\mathcal{D}^{\Gamma_\omega}\nabla_{\Gamma_\omega} u_\omega$, for a given symmetric random diffusion tensor $\mathcal{D}^\Gamma$ with realisations $\mathcal{D}^{\Gamma_\omega}(\cdot,t) : \Gamma_\omega(t) \to \mathbb{R}^{(n+1)\times(n+1)}$ at each time mapping $\mathcal{D}^{\Gamma_\omega}(\cdot,t) : T\Gamma_\omega(t) \to T\Gamma_\omega(t)$,

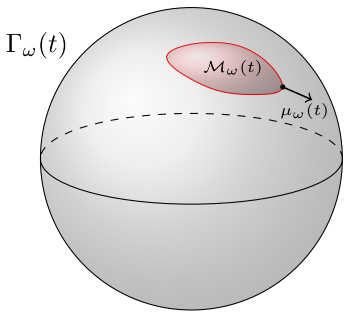

Figure 3.2: A conservation law on the randomly evolving surface Γ_ω formulated on an arbitrary portion $\mathcal{M}_\omega$ evolving by the random normal velocity field $v_{\nu^{\Gamma_\omega}}$.

and $\mathcal{D}^{\Gamma_\omega}(\cdot,t)\nu^{\Gamma_\omega}(\cdot,t) = 0$, leads to the following random surface-diffusion equation:

$$\partial^\circ u_\omega + u_\omega \nabla_{\Gamma_\omega} \cdot v_{\nu^{\Gamma_\omega}} - \nabla_{\Gamma_\omega} \cdot \left(\mathcal{D}^{\Gamma_\omega} \nabla_{\Gamma_\omega} u_\omega\right) = 0.$$

Such a random diffusion tensor may similarly to the case of a random tangential velocity field, be constructed by restricting onto the randomly evolving surface Γ, the projection of a matrix-valued random field $\tilde{D} : \Omega \times X \times [0,T] \to \mathbb{R}^{(n+1)\times(n+1)}$ onto the tangent space, i.e.

$$\mathcal{D}^{\Gamma_\omega}(\cdot,t) = \mathcal{P}_{\Gamma_\omega(t)} \tilde{D}(\omega;\cdot,t) \mathcal{P}_{\Gamma_\omega(t)}.$$

In many physical problems arising in biological systems [67] and fluid dynamics [53, 54], there is an additional physical advection of material points over the surface. For instance, when modelling the diffusion of an insoluble surfactant over the interface of a fluid, the concentration is further subjected to a convection driven by the tangential velocity of the fluid at the interface. We therefore may wish to further introduce a random advective flux $q_\omega^{adv} = v_{\tau,\omega} u_\omega$ over the surface Γ for a prescribed random tangential velocity field v_τ. Substituting in the additional flux $q_\omega = q_\omega^{dif} + q_\omega^{adv}$ into the pointwise conservation law (3.1.3) introduces to following the extra terms

$$\nabla_{\Gamma_\omega} \cdot q_\omega^{adv} = \nabla_{\Gamma_\omega} \cdot (v_{\tau,\omega} u_\omega) = \nabla_{\Gamma_\omega} u_\omega \cdot v_{\tau,\omega} + u_\omega \left(\nabla_{\Gamma_\omega} \cdot v_{\tau,\omega}\right),$$

and consequently, recalling that $\partial^\bullet_{v_\omega} u_\omega = \partial^\circ u_\omega + \nabla_{\Gamma_\omega} \cdot v_{\tau,\omega}$, leads to the following random advection-diffusion surface equation

$$\partial^\bullet_{v_\omega} u_\omega + u_\omega \nabla_{\Gamma_\omega} \cdot v_\omega - \nabla_{\Gamma_\omega} \cdot \left(\mathcal{D}^{\Gamma_\omega} \nabla_{\Gamma_\omega} u_\omega\right) = 0,$$

where $v_\omega = v_{\nu^{\Gamma_\omega}} + v_{\tau,\omega}$ denotes the random physical random velocity field of the surface Γ. This leads to our first model parabolic equation, in which we shall treat the specific case $\mathcal{D}^{\Gamma_\omega} = \mathcal{P}_{\Gamma_\omega}$.

Problem 3.1.1 (Advection-diffusion on a randomly evolving surface)**.** *For a.e. ω, find $u_\omega(\cdot,t) : \Gamma_\omega(t) \to \mathbb{R}$ such that*

$$\partial^\bullet_{v_\omega} u_\omega + u_\omega \nabla_{\Gamma_\omega} \cdot v_\omega - \Delta_{\Gamma_\omega} u_\omega = 0. \tag{3.1.4}$$

The concentration of the surface quantity may in many applications [12, 81], be further coupled with the concentration of a quantity within the bulk domain through an absorption/desorption process. For instance, when modelling the diffusion of membrane-bound proteins over a cell, the concentration is naturally coupled to the concentration of proteins diffusing within the inner cytoplasm due to the binding/dissociation mechanism in which the proteins attach/detach with the biomembrane. This subsequently leads to our second model problem, where we consider the case in which the physical random velocity field w within the randomly evolving bulk domain $D_\omega(t)$, coincides with the physical random surface velocity field.

Problem 3.1.2 (Coupled advection-diffusion on randomly evolving bulk-surface)**.** *For a.e. ω, find $(u_\omega(\cdot,t), v_\omega(\cdot,t)) : D_\omega(t) \times \Gamma_\omega(t) \to \mathbb{R}$ such that*

$$\partial^\bullet_{w_\omega} u_\omega + u_\omega \nabla \cdot w_\omega - \Delta u_\omega = 0 \quad \text{on } D_\omega(t) \tag{3.1.5a}$$

$$\alpha u_\omega - \beta v_\omega + \frac{\partial u_\omega}{\partial \nu_{\Gamma_\omega}} = 0 \quad \text{on } \Gamma_\omega(t) \tag{3.1.5b}$$

$$\partial^\bullet_{w_\omega} v_\omega + v_\omega \nabla_{\Gamma_\omega} \cdot w_\omega - \Delta_{\Gamma_\omega} v_\omega + \frac{\partial u_\omega}{\partial \nu_{\Gamma_\omega}} = 0 \quad \text{on } \Gamma_\omega(t). \tag{3.1.5c}$$

3.1.3 The extended domain mapping method

We now outline a general framework in which the domain mapping method may be extended to the case of randomly evolving domains. The main premise of the method to reformulate the original random domain problem onto a deterministic domain will remain the same. However, now the stochastic domain mapping will include a temporal component. To provide a framework in its full generality, we will allow for the deterministic reference domain to be time-dependent. This will lead to the incorporation of much wider range of random physical phenomema to be considered, such examples being an extension of [88], which investigates the effects of small random thermal fluctuations on lateral protein diffusion in a cell biomembrane to also include a long-term deterministic evolution of the cell. The general framework is as follows.

Computational Framework.

1. ***Computational domain:*** *We first select a computational deterministic reference domain which may or may not evolve in time. This may have some physical meaning for the problem, such as being the expected domain in the sense the expected value of the random boundary process is zero, or it may be selected in such a way that reduces the computational cost. For our purposes, we will assume that the reference domain*

$$\Gamma_0(t) \subset \mathbb{R}^{n+1} \quad t \in [0,T]$$

is a smooth, compact, deterministic hypersurface whose respective evolution from the initial surfaces $\Gamma_0(0)$ is completely characterised by the given normal velocity vector field $v_\nu^{\Gamma_0} = V^{\Gamma_0}\nu^{\Gamma_0}$.

2. ***Random boundary process:*** *We next characterise the randomly evolving surface by*

prescribing a stochastic mapping

$$\phi(t;\omega,\cdot) : \Gamma_0(t) \to \Gamma_\omega(t), \tag{3.1.6}$$

between the evolving reference surface and realisations of the random surface at each time, as illustrated in Figure 3.3. Naturally, the chosen parametrisation would have to satisify the following compatability condition

$$\left(\partial^\circ \phi \cdot (\nu^\Gamma \circ \phi)\right) \nu^\Gamma \circ \phi = v_\nu^\Gamma \circ \phi.$$

This ensures that the random velocity field associated to the flow map (3.1.6), which we shall denote by

$$v_{para}^\Gamma \circ \phi = \partial^\circ \phi, \tag{3.1.7}$$

coincides with the given random normal velocity of Γ *in the normal direction, i.e.* $(v_{para}^\Gamma \cdot \nu^\Gamma)\nu^\Gamma = v_\nu^\Gamma$*. However, the particular choice of the parametrisation will be treated as arbitrary, in the sense that we will not necessarily assume that it is given by the flow map of the random physical velocity field* $v^\Gamma = v_\nu^\Gamma + v_{\tau,phys}^\Gamma$ *appearing in the considered advection-diffusion surface equation. Thus instead, we impose no conditions on the tangential component in the associated velocity* v_{para}^Γ*,*

$$v_{para}^\Gamma = v_\nu^\Gamma + v_{\tau,para}^\Gamma,$$

which will be advantageous in many applications.

3. ***Extension and reformulation:*** *In the case of the randomly evolving bulk-surface system, the stochastic boundary process (3.1.6) will subsequently have to be extended into the interior bulk domain. There are numerous ways in which an extension may be defined, and depending on the application in question different options may be seleceted. For example, [101] propose an extension based upon the solution to the Laplace equation, alternatively [49] defines an extension with the help of a blending function in the normal direction to the random surface boundary. With the complete stochastic domain mapping constructed, we may now reformulate the original parabolic equations on the randomly evolving domain onto the deterministic computational domain.*

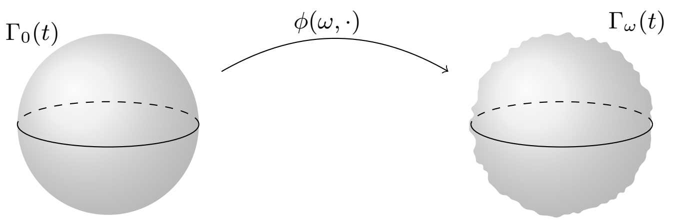

Figure 3.3: The evolving computational reference domain $\Gamma_0(t)$ and a realisation of the randomly evolving surface $\Gamma_\omega(t)$.

Remark 3.1.1 (Practical applications)**.** *Practical applications of the extended domain mapping method will fall into one of three categories. (1) The random physical velocity field of* Γ *is a-proiri known, in which case the stochastic boundary process (3.1.6) may be constructed by solving the corresponding stochastic ODE. (2) The random motion of the surface is governed by a stochastic geometric evolution equation. (3) The randomly evolving surface is initially prescribed by a stochastic mapping (3.1.6) with known statistical properties, such as the mean, and two-point (space-time) covariance function. In such a case, it is necessary to approximate the true random boundary process (3.1.6) over the evolving reference domain by an expansion in terms of a finite number of random variables, a common example being the Karhunen-Loeve decomposition. Questions relating to efficient and practical approximation of spatio-temporal random fields and random fields over surfaces has been an active area of research [6, 25, 43, 62, 84].*

3.2 Computations of time-dependent quantities on parametrised evolving surfaces

We now proceed by computing expressions for pull-back of time-dependent quantities on evolving surfaces which are parametrised over an evolving reference domain. Specifically, we compute the pull-back of the material derivative $\partial^{\bullet}_{v^\Gamma} u$ as well as the surface divergence $\nabla_\Gamma \cdot v^\Gamma$ of a general velocity field $v^\Gamma = v^\Gamma_\nu + v^\Gamma_\tau$ with an arbitrary tangential component v^Γ_τ, on the parametrised evolving surface. These quantities as previously discussed, arise naturally when considering parabolic equations in time-dependent domains; the material derivative being a suitable notion of the time-derivative and the divergence of a velocity field due to the expansion/contraction of the surface area element. Note that throughout the section, all the considered surfaces will be assumed to be deterministic.

3.2.1 The material derivative

To derive an expression for the pull-back of the material derivative $\partial^{\bullet}_{v^\Gamma} u$ onto the reference surface, we first need to compute the corresponding flow on $\Gamma_0(t)$ induced by the pull-back under $\phi(\cdot, t)$ of material points on $\Gamma(t)$ evolving by the given velocity v^Γ. We refer to the velocity field associated with the induced flow on $\Gamma_0(t)$ as the induced velocity $v^{\Gamma_0}_{ind,v^\Gamma}$ and define it more precisely as follows.

Definition 3.2.1 (Induced velocity on $\Gamma_0(t)$)**.** *The induced velocity* $v^{\Gamma_0}_{ind,v^\Gamma}$ *on* $\Gamma_0(t)$ *associated to the velocity* v^Γ *on* $\Gamma(t)$*, is defined by*

$$v^{\Gamma_0}_{ind,v^\Gamma}(y,t) = Y'(t) \quad y \in \Gamma_0(t),$$

where $\{Y(s)\}_{s\in[0,T]}$ *is the unique trajectory on* $\Gamma_0(t)$ *with* $Y(t) = y$*, which satisfies*

$$\phi(Y(s), s) = X(s) \quad X'(s) = v^\Gamma(X(s), s). \tag{3.2.1}$$

For convenience, we will often write $v_{ind}^{\Gamma_0}$ provided there is no ambiguity to which velocity field on $\Gamma(t)$ is refered to for the induced flow on $\Gamma_0(t)$. Since $v_{ind}^{\Gamma_0}$ describes a particular evolution of material points on the evolving surface $\Gamma_0(t)$ which has normal velocity $v_\nu^{\Gamma_0}$, it follows that it may be expressed as

$$v_{ind}^{\Gamma_0} = v_\nu^{\Gamma_0} + v_{\tau,corr}^{\Gamma_0}, \tag{3.2.2}$$

where $v_{\tau,corr}^{\Gamma_0}$ is a corrective tangential advection over $\Gamma_0(t)$ that appears due to the potential disagreement between the given parametrisation and the flow associated to v^Γ. We can compute the additional tangential velocity in the induced flow on $\Gamma_0(t)$ as follows.

Lemma 3.2.1 (Corrective advection on $\Gamma_0(t)$)**.** *The tangential component of the induced velocity field $v_{ind}^{\Gamma_0} = v_\nu^{\Gamma_0} + v_{\tau,corr}^{\Gamma_0}$ on $\Gamma_0(t)$ associated to $v^\Gamma = v_\nu^\Gamma + v_\tau^\Gamma$, is given by*

$$v_{\tau,corr}^{\Gamma_0} = G_{\Gamma_0}^{-1} \nabla_{\Gamma_0} \phi^\top \left(v_\tau^\Gamma \circ \phi - v_{\tau,para}^\Gamma \circ \phi \right), \tag{3.2.3}$$

where $v_{\tau,para}^\Gamma \circ \phi = (\mathcal{P}_\Gamma \circ \phi)\, \partial^\circ \phi$.

Proof. We observe by differentiating the relation (3.2.1) satisfied by trajectories on $\Gamma_0(t)$ evolving under the induced velocity $v_{ind}^{\Gamma_0}$, that the corrective tangential velocity satisfies

$$\partial^\circ \phi + \nabla_{\Gamma_0} \phi\, v_{\tau,corr}^{\Gamma_0} = \partial^\bullet_{v_{ind}^{\Gamma_0}} \phi = v^\Gamma \circ \phi.$$

Consequently, it follows that

$$\nabla_{\Gamma_0} \phi^\top \nabla_{\Gamma_0} \phi\, v_{\tau,corr}^{\Gamma_0} = \nabla_{\Gamma_0} \phi^\top \left(v^\Gamma \circ \phi - \partial^\circ \phi \right).$$

Since the velocity $v_{\tau,corr}^{\Gamma_0}$ is tangential on $\Gamma_0(t)$, we may extend the tensor $\nabla_{\Gamma_0}\phi^\top \nabla_{\Gamma_0}\phi$ in the normal direction to obtain the invertible matrix $G_{\Gamma_0} = \nabla_{\Gamma_0}\phi^\top \nabla_{\Gamma_0}\phi + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}$. Furthermore, due to the following orthogonality $\nabla_{\Gamma_0}\phi^\top \left(\nu^\Gamma \circ \phi \right) = 0$, we may additionally remove the normal components of each velocity v^Γ and $\partial^\circ \phi = v_{para} \circ \phi$, which combined leads to

$$G_{\Gamma_0}\, v_{\tau,corr}^{\Gamma_0} = \nabla_{\Gamma_0} \phi^\top \left(v_\tau^\Gamma \circ \phi - (\mathcal{P}_{\Gamma_0} \circ \phi)\, \partial^\circ \phi \right).$$

and hence the stated result.

$\square$

We may now give an explicit expression for the pull-back of the material derivative $\partial^\bullet_{v^\Gamma} u$ onto $\Gamma_0(t)$, as by the definition of the induced velocity, this is precisely given by $\left(\partial^\bullet_{v^\Gamma} u \right) \circ \phi = \partial^\bullet_{v_{ind}^{\Gamma_0}} (u \circ \phi)$.

Lemma 3.2.2 (Material derivative)**.** *Given any function $u(\cdot, t)$ defined on $\Gamma(t)$, let $\hat{u} = u \circ \phi$ denote the pull-back onto $\Gamma_0(t)$. Then the pull-back of the material derivative associated to v^Γ is given by*

$$\left(\partial^\bullet_{v^\Gamma} u \right) \circ \phi = \partial^\circ \hat{u} + \nabla_{\Gamma_0} \hat{u} \cdot v_{\tau,corr}^{\Gamma_0}. \tag{3.2.4}$$

3.2.2 Surface-divergence of a velocity field

We continue by computing the pull-back of the surface-divergence of a general velocity field $v^\Gamma = v^\Gamma_\nu + v^\Gamma_\tau$. For this, we first decompose v^Γ into velocity field associated with the given parametrisation and the additional corrective advection on $\Gamma(t)$,

$$v^\Gamma = v^\Gamma_{para} + \left(v^\Gamma_\tau - v^\Gamma_{\tau,para}\right). \tag{3.2.5}$$

The purpose of the above splitting is that the pull-back of the surface-divergence of the first term under the mapping $\phi(\cdot,t)$ will relate to the time-derivative of the surface area element $\sqrt{g_{\Gamma_0}}$. This is often seen in calculations in local coordinates as

$$\left(\nabla_{\Gamma_0} \cdot v^\Gamma_{para}\right) \circ X = \frac{1}{\sqrt{g}} \partial_t \sqrt{g}$$

where $g = det(\nabla X^\top \nabla X)$ and $X : U \to \mathbb{R}^{n+1}$ denoting a local parametrisation of the evolving surface over a flat (stationary) reference domain $U \subset \mathbb{R}^n$. We now generalise this result to the case of an evolving surface reference domain as follows.

Lemma 3.2.3 (Surface divergence of v^Γ_{para})**.** *The pull-back of the surface-divergence of the velocity field v^Γ_{para} corresponding the given parametrisation onto the evolving reference surface $\Gamma_0(t)$, is given by*

$$\left(\nabla_\Gamma \cdot v^\Gamma_{para}\right) \circ \phi = \frac{1}{\sqrt{g_{\Gamma_0}}} \partial^\circ \left(\sqrt{g_{\Gamma_0}}\right) + \nabla_{\Gamma_0} \cdot v^{\Gamma_0}_\nu. \tag{3.2.6}$$

Proof. Taking the normal-time derivative of the surface area element $\sqrt{g_{\Gamma_0}}$ with applications of the chain rule and the Jacobi's formula for the derivative of a determinant gives

$$\begin{aligned}\frac{1}{\sqrt{g_{\Gamma_0}}} \partial^\circ \left(\sqrt{g_{\Gamma_0}}\right) &= \frac{1}{2 g_{\Gamma_0}} \partial^\circ g_{\Gamma_0} \\ &= \frac{1}{2} trace\left(G^{-1}_{\Gamma_0} \partial^\circ G_{\Gamma_0}\right) = trace\left(G^{-1}_{\Gamma_0} \partial^\circ \left(\nabla_{\Gamma_0} \phi^\top\right) \nabla_{\Gamma_0} \phi + G^{-1}_{\Gamma_0} \left(\partial^\circ \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}\right)\right)\end{aligned}$$

where the last equality follows due to the symmetry of G_{Γ_0}. We next interchange the normal-time derivative and surface gradient with the identity

$$\partial^\circ \left(\nabla_{\Gamma_0} \phi\right) = \nabla_{\Gamma_0} \left(\partial^\circ \phi\right) + \nabla_{\Gamma_0} \phi \nabla_{\Gamma_0} V^{\Gamma_0} \otimes \nu^{\Gamma_0} - V^{\Gamma_0} \nabla_{\Gamma_0} \phi \mathcal{H}^{\Gamma_0},$$

recalling that V^{Γ_0} denotes the velocity of $\Gamma_0(t)$ in the normal direction, i.e. $v^{\Gamma_0}_\nu = V^{\Gamma_0} \nu^{\Gamma_0}$, to obtain

$$\begin{aligned}\frac{1}{\sqrt{g_{\Gamma_0}}} \partial^\circ \left(\sqrt{g_{\Gamma_0}}\right) &= trace\left(G^{-1}_{\Gamma_0} \nabla_{\Gamma_0} \left(\partial^\circ \phi\right)^\top \nabla_{\Gamma_0} \phi\right) + trace\left(G^{-1}_{\Gamma_0} \left(\nu^{\Gamma_0} \otimes \nabla_{\Gamma_0} \phi \nabla_{\Gamma_0} V^{\Gamma_0}\right) \nabla_{\Gamma_0} \phi\right) \\ &\quad - V^{\Gamma_0} trace\left(G^{-1}_{\Gamma_0} \mathcal{H}^{\Gamma_0} \nabla_{\Gamma_0} \phi^\top \nabla_{\Gamma_0} \phi\right) + trace\left(G^{-1}_{\Gamma_0} \left(\partial^\circ \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}\right)\right)\end{aligned}$$

Examining the second term, we have as a consequence of the following tensor product identities

$trace\,(a \otimes b) = a \cdot b$, $(a \otimes b)\, C = a \otimes C^\top b$, and the property $G_{\Gamma_0} \nu^{\Gamma_0} = \nu^{\Gamma_0}$ by construction, that

$$\mathrm{II} = \nu^{\Gamma_0} \cdot \left(\nabla_{\Gamma_0} \phi^\top \nabla_{\Gamma_0} \phi \nabla_{\Gamma_0} V^{\Gamma_0} \right) = \nabla_{\Gamma_0} \phi \, \nu^{\Gamma_0} \cdot \nabla_{\Gamma_0} \phi \nabla_{\Gamma_0} V^{\Gamma_0} = 0.$$

Similarly, we observe with the identity $\partial^\circ \nu^{\Gamma_0} = -\nabla_{\Gamma_0} V^{\Gamma_0}$ that the fourth term also vanishes. For the third term, we recall that g_{jk} denote the entries of G_{Γ_0} and g^{jk} the entries of its inverse and furthermore $\mathcal{H}^{\Gamma_0} \nu^{\Gamma_0} = 0$, to deduce

$$\mathrm{III} = -V^{\Gamma_0} \sum_{j,k,m} g^{jk} \mathcal{H}^{km} \left(g_{mj} - \nu_m^{\Gamma_0} \nu_j^{\Gamma_0} \right) = -V^{\Gamma_0}\, trace \left(\mathcal{H}^{\Gamma_0} \right) = -V^{\Gamma_0}\, \nabla_{\Gamma_0} \cdot v_\nu^{\Gamma_0}.$$

The last step follows by observing

$$\nabla_{\Gamma_0} \cdot v_\nu^{\Gamma_0} = \nabla_{\Gamma_0} V^{\Gamma_0} \cdot \nu^{\Gamma_0} + V^{\Gamma_0} \nabla_{\Gamma_0} \cdot \nu^{\Gamma_0} = V^{\Gamma_0}\, trace \left(\mathcal{H}^{\Gamma_0} \right).$$

Finally, with the pull-back of the surface divergence given by

$$\nabla_\Gamma \cdot f = trace \left(G_{\Gamma_0}^{-1} \nabla_{\Gamma_0} \phi^\top \nabla_{\Gamma_0} \left(f \circ \phi \right) \right)$$

and $v_{para}^\Gamma \circ \phi = \partial^\circ \phi$, we deduce $\mathrm{I} = \left(\nabla_\Gamma \cdot v_{para}^\Gamma \right) \circ \phi$ and therefore combining the results we obtain the stated result. $\square$

The second term in (3.2.5), corresponding to the corrective advection over $\Gamma(t)$ will now be treated by computing the pull-back of $\nabla_\Gamma \cdot v_\tau^\Gamma$ for a general tangential velocity v_τ^Γ and after which applying the result to the specific advection under consideration. In order to simplify the computations, we introduce a generalisation of the Christoffel symbols of the second kind for parametrised surfaces where the reference domain is additionally taken to be a surface.

Definition 3.2.2 (Generalised Christoffel symbols)**.** *We define*

$$\Gamma_{ij}^k = \underline{D}_j^{\Gamma_0} \underline{D}_i^{\Gamma_0} \phi \cdot \underline{D}_k^{\Gamma_0} \phi \qquad i, j, k = 1, ..., n+1. \tag{3.2.7}$$

and interpret these values as the coefficients of the tangential component of $\underline{D}_j^{\Gamma_0} \underline{D}_i^{\Gamma_0} \phi$ with respect to the spanning set of vectors $\{\underline{D}_k^{\Gamma_0}\}_k \subset T_{\phi(\cdot)}\Gamma$,

$$\underline{D}_j^{\Gamma_0} \underline{D}_i^{\Gamma_0} \phi = \sum_k \Gamma_{ij}^k \underline{D}_k^{\Gamma_0} \phi + \left(\underline{D}_j^{\Gamma_0} \underline{D}_i^{\Gamma_0} \phi \cdot \left(\nu^\Gamma \circ \phi \right) \right) \nu^\Gamma \circ \phi. \tag{3.2.8}$$

Here recall $\underline{D}_j^{\Gamma_0} \phi$ are tangential due to the fact that the restricition $\nabla_{\Gamma_0} \phi : T\Gamma_0 \to T_{\phi(\cdot)}\Gamma$ is a bijective mapping. We note the generalised Christoffel symbols satisfy the following properties:

$$\sum_l \Gamma_{ij}^l \nu_l^{\Gamma_0} = 0 \qquad \Gamma_{ij}^k = \sum_l \Gamma_{ij}^l g_{lk}. \tag{3.2.9}$$

The first identity immediately follows from the orthoganality $\sum_m \underline{D}_l^{\Gamma_0} \phi_m \nu_l^{\Gamma_0} = 0$. The second result may be observed by substituting in the representation of $\underline{D}_j^{\Gamma_0}\underline{D}_i^{\Gamma_0}\phi$ given in (3.2.8) into the definition (3.2.7), cancelling the normal term due the orthoganality $\left(\nu^{\Gamma}\circ\phi\right)\cdot\underline{D}_l^{\Gamma_0}\phi$,

$$\Gamma_{ij}^k = \sum_l \Gamma_{ij}^l\left(\underline{D}_l^{\Gamma_0}\phi\cdot\underline{D}_k^{\Gamma_0}\phi\right) = \sum_l \Gamma_{ij}^l\left(g_{lk} - \nu_l^{\Gamma_0}\nu_k^{\Gamma_0}\right) = \sum_l \Gamma_{ij}^l g_{lk}$$

and applying the previous result. We may now compute the pull-back of the surface-divergence for a general tangential velocity as follows.

Lemma 3.2.4 (Surface-divergence of v_τ^Γ)**.** *Given an arbitrary tangential velocity field v_τ^Γ over the evolving surface $\Gamma(t)$, the pull-back of the surface-divergence is given by*

$$\left(\nabla_\Gamma\cdot v_\tau^\Gamma\right)\circ\phi = \frac{1}{\sqrt{g_{\Gamma_0}}}\nabla_{\Gamma_0}\cdot\left(\sqrt{g_{\Gamma_0}}G_{\Gamma_0}^{-1}\nabla_{\Gamma_0}\phi^\top\left(v_\tau^\Gamma\circ\phi\right)\right).$$

Proof. We express the given tangential velocity v_τ^Γ in terms of the spanning set $\{\underline{D}_j^{\Gamma_0}\phi\}\subset T_{\phi(\cdot)}\Gamma$ as

$$v_\tau^\Gamma\circ\phi = \sum_{i=1}^{n+1}\alpha_i\underline{D}_i^{\Gamma_0}\phi = \nabla_{\Gamma_0}\phi\,\alpha, \tag{3.2.10}$$

for a unique tangential vector $\alpha = (\alpha_i)_{i=1}^{n+1}\in T_{\phi(\cdot)}\Gamma$. By a similar argument as previously seen, we may compute the coefficients α by multiplying (3.2.10) by $\nabla_{\Gamma_0}\phi^\top$ and expanding the tensor $\nabla_{\Gamma_0}\phi^\top\nabla_{\Gamma_0}\phi$ in the normal direction to the obtain the invertible matrix G_{Γ_0}, giving

$$\alpha = G_{\Gamma_0}^{-1}\nabla_{\Gamma_0}\phi^\top\left(v_\tau^\Gamma\circ\phi\right).$$

Thus, it remains to prove $\left(\nabla_\Gamma\cdot v_\tau^\Gamma\right)\circ\phi = \frac{1}{\sqrt{g_{\Gamma_0}}}\nabla_{\Gamma_0}\cdot\left(\sqrt{g_{\Gamma_0}}\alpha\right)$. Examining the divergence term first, we apply the identity

$$\left(\nabla_\Gamma\cdot v_\tau^\Gamma\right)\circ\phi = \sum_{j,k} g^{jk}\underline{D}_j^{\Gamma_0}\left(v_\tau^{\Gamma_0}\circ\phi\right)\cdot\underline{D}_k^{\Gamma_0}\phi,$$

and substitute in the representation of the tangential velocity given in (3.2.10), noting that $g_{ik} = \underline{D}_i^{\Gamma_0}\phi\cdot\underline{D}_k^{\Gamma_0}\phi + \nu_i^{\Gamma_0}\nu_k^{\Gamma_0}$, to obtain

$$\begin{aligned}\left(\nabla_\Gamma\cdot v_\tau^\Gamma\right)\circ\phi &= \sum_{i,j,k} g^{jk}\left(\underline{D}_j^{\Gamma_0}\alpha_i\underline{D}_i^{\Gamma_0}\phi + \alpha_i\underline{D}_j^{\Gamma_0}\underline{D}_i^{\Gamma_0}\phi\right)\cdot\underline{D}_k^{\Gamma_0}\phi\\ &= \sum_{i,j,k} g^{jk}\underline{D}_j^{\Gamma_0}\alpha_i\left(g_{ik} - \nu_i^{\Gamma_0}\nu_k^{\Gamma_0}\right) + \sum_{i,j,k} g^{jk}\alpha_i\Gamma_{ij}^k\\ &= \nabla_{\Gamma_0}\cdot\alpha + \sum_{i,l}\Gamma_{il}^l\alpha_i.\end{aligned}$$

Here for the last step, we observe that the unit normal term vanishes due to the first identity (3.2.9) satisfied by the Christoffel symbols. Furthermore, the second summation simplifies as indicated as a consequence of the second identity satisfied by the Christoffel symbols. We next consider

$$\frac{1}{\sqrt{g_{\Gamma_0}}}\nabla_{\Gamma_0}\cdot\left(\sqrt{g_{\Gamma_0}}\alpha\right)=\nabla_{\Gamma_0}\cdot\alpha+\frac{1}{\sqrt{g_{\Gamma_0}}}\alpha\cdot\nabla_{\Gamma_0}\left(\sqrt{g_{\Gamma_0}}\right).$$

Differentiating the surface area element $\sqrt{g_{\Gamma_0}}$, with the Jacobi formula for the derivative of a determinant yields

$$\frac{1}{\sqrt{g_{\Gamma_0}}}\alpha\cdot\nabla_{\Gamma_0}\left(\sqrt{g_{\Gamma_0}}\right)=\frac{1}{2g_{\Gamma_0}}\sum_i\alpha_i\underline{D}_i^{\Gamma_0}g_{\Gamma_0}=\frac{1}{2}\sum_i trace\left(G_{\Gamma_0}^{-1}\underline{D}_i^{\Gamma_0}G_{\Gamma_0}\right)=\frac{1}{2}\sum_{i,j}g^{jk}\underline{D}_i^{\Gamma_0}\left(g_{ik}\right)\alpha_i.$$

We next differentiate the metric $g^{jk}=\underline{D}_k^{\Gamma_0}\phi\cdot\underline{D}_j^{\Gamma_0}\phi+\nu_k^{\Gamma_0}\nu_j^{\Gamma_0}$ which leads to

$$=\frac{1}{2}\sum_{i,j,k}g^{jk}\left(\Gamma^j_{ki}+\Gamma^k_{ji}+\mathcal{H}_{ik}^{\Gamma_0}\nu_j^{\Gamma_0}+\mathcal{H}_{ij}^{\Gamma_0}\nu_k^{\Gamma_0}\right)\alpha_i$$

Both of the normal components appearing above vanish due to the orthoganality $\sum g^{ik}\mathcal{H}_{ik}^{\Gamma_0}\nu_j^{\Gamma_0}=\sum\mathcal{H}_{ik}^{\Gamma_0}\nu_k^{\Gamma_0}=0$. We continue by substituting in the given indentity (3.2.9) for Γ^j_{ki} which yields

$$=\frac{1}{2}\sum_{i,j,k,l}g^{jk}\left(\Gamma^l_{ki}g_{lj}+\Gamma^l_{ji}g_{lk}\right)\alpha_i=\sum_{i,l}\Gamma^l_{li}\alpha_i. \tag{3.2.11}$$

Finally, it remains to show that we may interchange Γ^l_{li} with Γ^l_{il} in the above summation. For this, we observe by interchanging the tangential derivatives

$$\Gamma^l_{li}-\Gamma^l_{il}=\left(\underline{D}_l^{\Gamma_0}\underline{D}_i^{\Gamma_0}\phi-\underline{D}_i^{\Gamma_0}\underline{D}_l^{\Gamma_0}\phi\right)\cdot\underline{D}_l^{\Gamma_0}\phi=\sum_k\left(\nu_l^{\Gamma_0}\mathcal{H}_{ik}^{\Gamma_0}\underline{D}_k^{\Gamma_0}-\nu_i^{\Gamma_0}\mathcal{H}_{lk}^{\Gamma_0}\underline{D}_k^{\Gamma_0}\phi\right)\cdot\underline{D}_l^{\Gamma_0}\phi,$$

which simplifies to give

$$=\sum_k\left(\nu_l^{\Gamma_0}\mathcal{H}_{ik}^{\Gamma_0}-\nu_i^{\Gamma_0}\mathcal{H}_{lk}^{\Gamma_0}\right)\left(\underline{D}_k^{\Gamma_0}\phi\cdot\underline{D}_l^{\Gamma_0}\phi\right)=-\sum_k\nu_i^{\Gamma_0}\mathcal{H}_{lk}^{\Gamma_0}\left(\underline{D}_k^{\Gamma_0}\phi\cdot\underline{D}_l^{\Gamma_0}\phi\right),$$

where we have cancelled the orthogonal term due to $\sum_l\nu_l^{\Gamma_0}\underline{D}_l^{\Gamma_0}\phi=0$. Therefore, after we interchange Γ^l_{li} with Γ^l_{il} in the summation (3.2.11), we obtain and an additional term given above. However as α is tangential, this further term will vanish when summing $\alpha_i\nu_i^{\Gamma_0}$. □

Combining the results of Lemma 3.2.3 and Lemma 3.2.4 to the previous decomposition of the divergence of the given velocity $v^\Gamma=v^\Gamma_\nu+v^\Gamma_\tau$,

$$\nabla_\Gamma\cdot v^\Gamma=\nabla_\Gamma\cdot v^\Gamma_{para}+\nabla_\Gamma\cdot\left(v^\Gamma_\tau-v^\Gamma_{\tau,para}\right),$$

where the pull-back of the corrective advection simplifies as

$$\begin{aligned}&\left(\nabla_\Gamma\cdot\left(v^\Gamma_\tau-v^\Gamma_{\tau,para}\right)\right)\circ\phi\\&\quad=\frac{1}{\sqrt{g_{\Gamma_0}}}\nabla_{\Gamma_0}\cdot\left(\sqrt{g_{\Gamma_0}}G_{\Gamma_0}^{-1}\nabla_{\Gamma_0}\phi^\top\left(v^\Gamma_\tau\circ\phi-v^\Gamma_{\tau,para}\circ\phi\right)\right)=\frac{1}{\sqrt{g_{\Gamma_0}}}\nabla_{\Gamma_0}\cdot\left(\sqrt{g_{\Gamma_0}}v^{\Gamma_0}_{\tau,corr}\right)\end{aligned}$$

leads to the following expression.

Corollary 3.2.1 (Surface-divergence of v^Γ)**.** *The pull-back of the surface divergence of the given velocity $v^\Gamma = v^\Gamma_\nu + v^\Gamma_\tau$ over $\Gamma(t)$ is given by*

$$\left(\nabla_\Gamma \cdot v^\Gamma\right) \circ \phi = \frac{1}{\sqrt{g_{\Gamma_0}}} \partial^\circ \left(\sqrt{g_{\Gamma_0}}\right) + \nabla_{\Gamma_0} \cdot v_\nu^{\Gamma_0} + \frac{1}{\sqrt{g_{\Gamma_0}}} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}}\, v^{\Gamma_0}_{\tau,corr}\right), \tag{3.2.12}$$

where the corrective tangential advection $v^{\Gamma_0}_{\tau,corr}$ over $\Gamma_0(t)$ is given as in (3.2.3).

Corollary 3.2.2 (Pointwise rate of change of mass)**.** *Given a function $u(\cdot,t) : \Gamma(t) \to \mathbb{R}$ with associated pull-back $\hat{u} = u \circ \phi$ on $\Gamma_0(t)$, the pull-back of the rate of change of mass under the transport v^Γ is given by*

$$\left(\partial^\bullet_{v^\Gamma} u + u \nabla_\Gamma \cdot v^\Gamma\right) \circ \phi = \frac{1}{\sqrt{g_{\Gamma_0}}} \left(\partial^\circ \left(\sqrt{g_{\Gamma_0}} \hat{u}\right) + \sqrt{g_{\Gamma_0}} \hat{u} \nabla_{\Gamma_0} \cdot v_\nu^{\Gamma_0} + \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}} \hat{u} v^{\Gamma_0}_{\tau,corr}\right)\right).$$

Proof. Combining the pull-back of the material derivative associated to v^Γ

$$\left(\partial^\bullet_{v^\Gamma} u\right) \circ \phi = \partial^\circ \hat{u} + \nabla_{\Gamma_0} \hat{u} \cdot v^{\Gamma_0}_{\tau,corr}$$

given in Lemma 3.2.2 with Corollary 3.2.1 leads to

$$\begin{aligned}\left(\partial^\bullet_{v^\Gamma} u + u \nabla_\Gamma \cdot v^\Gamma\right) \circ \phi &= \frac{1}{\sqrt{g_{\Gamma_0}}} \left(\sqrt{g_{\Gamma_0}} \partial^\circ \hat{u} + \hat{u} \partial^\circ \left(\sqrt{g_{\Gamma_0}}\right) + \sqrt{g_{\Gamma_0}} \hat{u} \nabla_{\Gamma_0} \cdot v_\nu^{\Gamma_0}\right) \\ &\quad + \frac{1}{\sqrt{g_{\Gamma_0}}} \left(\hat{u} \nabla_{\Gamma_0} \cdot \left(\sqrt{g_{\Gamma_0}} v^{\Gamma_0}_{\tau,corr}\right) + \sqrt{g_{\Gamma_0}} \nabla_{\Gamma_0} \hat{u} \cdot v^{\Gamma_0}_{\tau,corr}\right)\end{aligned}$$

and subsequently the stated result. □

3.3 Abstract random solution spaces on evolving domains and their tensor structure

The purpose of this section is to introduce in a general abstract setting, the random function spaces that will be required in order to analyse as well as provide a rigorous mean-weak formulation for the random PDEs on deterministically evolving domains which arise after the extended domain mapping method has been applied. We will first begin by providing a brief overview into the generalisation of standard Bochner spaces $L^2(0,T;H)$ to the case of time-dependent Hilbert spaces $H(t)$, as proposed in [2, 95], which will be necessary in the analysis of any PDE on an evolving domain. It is worth noting that the construction of these spaces is reliant on a Lagrangian formulation, using a parametrisation defined on the initial domain, which will be well-suited for our evolving finite element discretisation. We will then proceed by defining the evolving random function spaces e.g. $L(0,T;L^2(\Omega, H(t)))$, appropriate for the reformulated random PDEs, and will prove that if all the necessary assumptions are satisfied to ensure the deterministic evolving function spaces e.g. $L^2(0,T;H(t))$ are well-defined, then the same assumptions for the equivalent random function spaces e.g. $L^2(0,T;L^2(\Omega,H(t)))$ also hold and therefore the random function spaces are well-defined.

3.3.1 Evolving Sobolev-Bochner spaces

The general setting for an evolving Sobolev-Bochner space which will serve as a solution space for a parabolic equation on an evolving domain is as follows. We have at each time $t \in [0,T]$, a Gelfand triple

$$V(t) \subset H(t) \subset V^*(t)$$

of separable real Hilbert spaces, whose evolution in time is prescribed by a given push-forward operator

$$\phi_t : H_0 \to H(t) \quad \phi_t|_{V_0} : V_0 \to V(t).$$

Here, we have denoted the initial Hilbert triple by $V_0 \subset H_0 \subset V_0^*$. The given push-forward operator ϕ_t, which in applications is a mapping related to the evolution of the reference domain from its initial configuration, will be assumed to satisfy the following minimal conditions listed in [2].

Assumption 3.3.1 (Evolving reference domain)**.** *We assume the push-forward operator ϕ_t : $H_0 \to H(t)$ and its restriction $\phi_t|_{V_0} : V_0 \to V(t)$ are both homeomorphisms and futhermore that there exists constants $C_1, C_2 > 0$ independent of t, such that*

$$C_1\|u_0\|_{H_0} \le \|\phi_t u_0\|_{H(t)} \le C_2\|u_0\|_{H_0} \quad \forall u_0 \in H_0$$
$$C_1\|u_0\|_{V_0} \le \|\phi_t u_0\|_{V(t)} \le C_2\|u_0\|_{V_0} \quad \forall u_0 \in V_0.$$

Additionally, we assume both of the following mappings are continuous

$$t \mapsto \|\phi_t u_0\|_{H(t)} \quad \text{for } u_0 \in H_0, \qquad t \mapsto \|\phi_t u_0\|_{V(t)} \quad \text{for } u_0 \in V_0.$$

Here, we denote the inverse of both mapping without ambiguity by ϕ_{-t}. The evolving Lebesgue-Bochner spaces are then defined by the given isomorphism with the standard Lebsesgue-Bochner spaces taking values in the initial Hilbert space,

$$L^2_V = \{u = \phi_t u_0 \,|\, u_0 \in L^2(0,T;V_0)\}$$
$$L^2_H = \{u = \phi_t u_0 \,|\, u_0 \in L^2(0,T;H_0)\}$$
$$L^2_{V^*} = \{f = \phi^*_{-t} f_0 \,|\, f_0 \in L^2(0,T;V_0^*)\},$$

where ϕ^*_{-t} denotes the dual operator to the mapping $\phi_{-t} : V(t) \to V_0$. These spaces are equipped with the following inner product

$$(u,v)_{L^2_X} = \int_0^T (u(t),v(t))_{X(t)}\,dt \qquad X = V, H, V^*,$$

to form separable Hilbert spaces. We next introduce a notion of evolving function spaces which are smooth in time, with the help of the prescribed isomorphism $\phi_t : X_0 \to X(t)$ between the Hilbert space $X(t)$ and the initial Hilbert space, with $X = H, V$.

Definition 3.3.1 (Smooth evolving spaces). *For $k \in \mathbb{N}$, we define the spaces*

$$
\begin{aligned}
C_X^k &= \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C^k([0,T]; X_0)\} && (3.3.1)\\
\mathcal{D}_X(0,T) &= \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C_0^\infty((0,T); X_0)\} && (3.3.2)\\
\mathcal{D}_X[0,T] &= \{\xi \in L_X^2 \mid \phi_{-(\cdot)}\xi(\cdot) \in C^\infty([0,T]; X_0)\}. && (3.3.3)
\end{aligned}
$$

Functions belonging to such spaces possess a natural definition of a material derivative, as the time derivative of the pull-back of the function under the prescribed flow map ϕ_t. More precisely, for $\eta \in C_X^1$ we define the strong material derivative $\partial^\bullet \eta \in C_X^0$ by

$$\partial^\bullet \eta = \phi_t \left(\frac{d}{dt} \phi_{-t} \eta(t) \right). \tag{3.3.4}$$

A weaker notion of a material derivative may also be defined based upon on the following transport property

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = (\partial^\bullet u(t), v(t))_{H(t)} + (u(t), \partial^\bullet v(t))_{H(t)} + \lambda(t; u(t), v(t)) \qquad \forall u, v \in C_H^1,$$

where the additional bilinear form $\lambda(t;\cdot,\cdot) : H(t) \times H(t) \to \mathbb{R}$ arising due to the time-dependency of the Hilbert space $H(t)$, is given by

$$
\begin{aligned}
\lambda(t; u, v) &= \hat{\lambda}(t; \phi_{-t}u, \phi_{-t}v) && (3.3.5)\\
\hat{\lambda}(t; u_0, v_0) &= \frac{d}{dt}(\phi_t u_0, \phi_t v_0)_{H(t)} \quad \forall u_0, v_0 \in H_0. && (3.3.6)
\end{aligned}
$$

This subsequently leads to the following definition of a weak material derivative.

Definition 3.3.2 (Weak material derivative). *Given $u \in L_V^2$, the weak material derivative $\partial^\bullet u \in L_{V^*}^2$ is uniquely defined by the relation*

$$\int_0^T \langle \partial^\bullet u(t), v(t) \rangle_{V^*(t),V(t)} = -\int_0^T (u(t), \partial^\bullet v(t))_{H(t)} - \int_0^T \langle \Lambda(t)u(t), v(t) \rangle_{H^*(t),H(t)} \tag{3.3.7}$$

for all $v \in \mathcal{D}_V(0,T)$.

The evolving Sobolev-Bochner spaces are then defined by

$$W(V, V^*) = \{u \in L_V^2 \mid \partial^\bullet u \in L_{V^*}^2\} \quad W(V, H) = \{u \in L_V^2 \mid \partial^\bullet u \in L_H^2\}.$$

For more details on these spaces, including further discussion of a weak-material derivative and the assumptions on the push-forward operator ϕ_t, see A.1.

3.3.2 Evolving random Sobolev-Bochner spaces

Let us now return to our original problem of defining suitable evolving function spaces to analyse the mean-weak formulation of the resulting random PDEs over deterministically evolving domains which arise from the extended domain mapping method. The appropriate evolving Bochner space will comprise of functions which at each $t \in [0,T]$ belong to a random function space defined over the domain at the given time, e.g. $u(t) \in L^2(\Omega; L^2(D(t)))$ with $D(t)$ denoting the evolving reference domain. In an abstract notation, the setting for these spaces will be as follows. We will have at each $t \in [0,T]$ a Gelfand triple

$$\mathcal{V}(t) \subset \mathcal{H}(t) \subset \mathcal{V}^*(t)$$

of separable Hilbert spaces and will define the random function spaces by

$$V(t) = L^2(\Omega; \mathcal{V}(t)) \quad H(t) = L^2(\Omega; \mathcal{H}(t)) \quad V^*(t) = L^2(\Omega; \mathcal{V}^*(t)).$$

Since the dual space of $L^2(\Omega; \mathcal{V}(t))$ is precisely given by $L^2(\Omega; \mathcal{V}^*(t))$, it follows that

$$V(t) \subset H(t) \subset V^*(t)$$

also forms a Gelfand triple at each $t \in [0,T]$. Furthermore, as we assumed that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is separable, it follows that $L^2(\Omega)$ is a separable space, see A.3.1. Hence by Theorem A.3.2, all of the considered random function spaces $V(t), H(t), V^*(t)$ admit a tensor structure

$$\begin{aligned} L^2(\Omega; \mathcal{V}(t)) &\cong L^2(\Omega) \otimes \mathcal{V}(t) \\ L^2(\Omega; \mathcal{H}(t)) &\cong L^2(\Omega) \otimes \mathcal{H}(t) \\ L^2(\Omega; \mathcal{V}^*(t)) &\cong L^2(\Omega) \otimes \mathcal{V}^*(t) \end{aligned}$$

and are therefore separable. We are now in the desired setting to formulate the evolving Bochner spaces $L^2_V, L^2_H, L^2_{V^*}$. For the push-forward operator, we assume we already have a mapping

$$\phi_t : \mathcal{H}_0 \to \mathcal{H}(t) \qquad \phi_t|_{\mathcal{V}_0} : \mathcal{V}_0 \to \mathcal{V}(t)$$

at hand for the deterministic function spaces, such that both the pairs $(\mathcal{H}, \phi_t)$ and $(\mathcal{V}, \phi_t)$ are compatible. In practice, this will equate to assuming that the evolution of the deterministic reference domain is sufficiently smooth, see A.1.2. We may then define an extension of the push-forward operator onto the random function spaces

$$\phi_t : H_0 \to H(t) \qquad \phi_t|_{V_0} : V_0 \to V(t)$$

by defining the mapping pathwise

$$(\phi_t u_0)(\omega) = \phi_t(u_0(\omega)) \qquad \forall u_0 \in H_0 = L^2(\Omega; \mathcal{H}(0)).$$

The compatibility of the pairs (H, ϕ_t) and $(V, \phi_t|_{V_0})$ will immediately follow from the assumed compatibility of the deterministic pairs $(\mathcal{H}, \phi_t)$ and $(\mathcal{V}, \phi_t)$. This may be observed by noting that all the constants $C_i > 0$ appearing in the estimates

$$C_1\|u_0(\omega)\|_{\mathcal{H}_0} \leq \|\phi_t u_0(\omega)\|_{\mathcal{H}(t)} \leq C_2\|u_0(\omega)\|_{\mathcal{H}_0} \quad \forall u_0 \in \mathcal{H}_0$$
$$C_1\|u_0(\omega)\|_{\mathcal{V}_0} \leq \|\phi_t u_0(\omega)\|_{\mathcal{V}(t)} \leq C_2\|u_0(\omega)\|_{\mathcal{V}_0} \quad \forall u_0 \in \mathcal{V}_0$$

are independent of ω. We summarise the results into the following lemma for future reference.

Lemma 3.3.1 (Compatability of random spaces)**.** *Let $(\mathcal{H}, \phi_t)$ and $(\mathcal{V}, \phi_t|_{\mathcal{V}_0})$ denote compatible pairs. Then their the natural extension to random function spaces $(L^2(\Omega; \mathcal{H}), \phi_t)$ and $(L^2(\Omega; \mathcal{V}), \phi_t|_{L^2(\Omega;\mathcal{V}_0)})$ with the push-forward operator defined path-wise, also defines compatible pairs.*

The evolving random Bochner spaces L^2_V, L^2_H and $L^2_{V^*}$ are now well-defined spaces and are isomorphic by construction with their respective initial Bochner spaces

$$L^2_X \cong L^2(0, T; X_0) = L^2(0, T; L^2(\Omega, \mathcal{X}_0) \qquad X = V, H, V^*.$$

Observing that as all of the considered Hilbert spaces $L^2([0, T]), L^2(\Omega)$ and $\mathcal{X}_0$ are separable spaces, we may exploit the tensor strucure of the initial Bochner space provided in Theorem A.3.2, to deduce that

$$L^2(0, T; L^2(\Omega, \mathcal{X}_0)) \cong L^2([0, T]) \otimes L^2(\Omega) \otimes \mathcal{X}_0 \cong L^2(\Omega; L^2(0, T; \mathcal{X}_0)). \tag{3.3.8}$$

We therefore obtain the following isomorphism for our evolving random Bochner spaces, by pushing forward the function space onto the $\mathcal{X}(t)$ with push-forward operator defined pathwise.

Theorem 3.3.1 (Isomorphism)**.** *The evolving random Bochner spaces $L^2_V, L^2_H, L^2_{V^*}$ are isometrically isomorphic to the following spaces*

$$L^2_V \cong L^2(\Omega; L^2_{\mathcal{V}})$$
$$L^2_H \cong L^2(\Omega; L^2_{\mathcal{H}})$$
$$L^2_{V^*} \cong L^2(\Omega; L^2_{\mathcal{V}^*}).$$

Remark 3.3.1. *Hence, we may equivalently consider functions belonging to the evolving random Bochner space $f \in L^2_H$ as a mapping at each t to a random function space $f(t) \in L^2(\Omega; \mathcal{H}(t))$, or as a random field with realisations defining functions over the evolving domain $f(\omega) \in L^2_{\mathcal{H}}$. We will often write*

$$f(t, \omega) = f(t)(\omega)$$

which is justified by the stated tensor structure (3.3.8) of the pull-back of the evolving random Bochner space L^2_H.

We continue by identifying the evolving random Sobolev-Bochner spaces $W(V,V^*)$ and $W(V,H)$ with the following isomorphisms

$$W(V,V^*) \cong L^2(\Omega; W(\mathcal{V},\mathcal{V}^*)) \tag{3.3.9}$$

$$W(V,H) \cong L^2(\Omega; W(\mathcal{V},\mathcal{H})). \tag{3.3.10}$$

This will be achieved by showing that realisations of the weak material derivative of a function $f \in L^2_V$ as defined in Definition 3.3.2, are precisely given by the weak material derivative of $f_\omega \in L^2_{\mathcal{V}}$ with respect to the Hilbert triple $(\mathcal{V}(t), \mathcal{H}(t), \mathcal{V}^*(t))$. Let us first begin by observing that the strong material derivative of a function belonging to the evolving random Bochner space $\eta \in C^1_X$ for $X = V, H$ is precisely given by the strong material derivative of its realisations

$$(\partial^\bullet \eta)(t,\omega) = (\partial^\bullet \eta_\omega)(t).$$

This is a direct consequence of the push-forward operator being defined path-wise. We therefore obtain the following isomorphism for the space of smooth in time evolving functions

$$C^1_X \cong L^2(\Omega; C^1_{\mathcal{X}}) \qquad X = H, V. \tag{3.3.11}$$

We next consider the extra term arising in the transport property

$$\frac{d}{dt}(\sigma_1,\sigma_2)_{H(t)} = (\partial^\bullet\sigma_1,\sigma_2)_{H(t)} + (\sigma_1,\partial^\bullet\sigma_2)_{H(t)} + \lambda(t;\sigma_1,\sigma_2) \qquad \forall \sigma_1,\sigma_2 \in C^1_H,$$

defined by

$$\lambda(t;\sigma_1,\sigma_2) = \hat{\lambda}(t;\phi_{-t}\sigma_1,\phi_{-t}\sigma_2)$$

with $\hat{\lambda}(t;\cdot,\cdot) : H_0(t) \times H_0(t) \to \mathbb{R}$ given by

$$\hat{\lambda}(t;\hat{\sigma}_1,\hat{\sigma}_2) = \frac{d}{dt}(\phi_t\hat{\sigma}_1,\phi_t\hat{\sigma}_2)_{H(t)} = \frac{d}{dt}\int_\Omega (\phi_t\hat{\sigma}_1(\omega),\phi_t\hat{\sigma}_2(\omega))_{\mathcal{H}(t)}.$$

Here we observe that this extra term $\lambda(t;\cdot,\cdot) : H(t) \times H(t) \to \mathbb{R}$, is precisely given by

$$\lambda(t;\sigma_1,\sigma_2) = \int_\Omega \tilde{\lambda}(t;\sigma_1(\omega),\sigma_2(\omega)) \tag{3.3.12}$$

the expectation of the extra term $\tilde{\lambda}(t;\cdot,\cdot) : \mathcal{H}(t) \times \mathcal{H}(t) \to \mathbb{R}$ appearing in the deterministic transport property

$$\frac{d}{dt}(\sigma_1,\sigma_2)_{\mathcal{H}(t)} = (\partial^\bullet\sigma_1,\sigma_2)_{\mathcal{H}(t)} + (\sigma_2,\partial^\bullet\sigma_2)_{\mathcal{H}(t)} + \tilde{\lambda}(t;\sigma_1,\sigma_2) \qquad \forall \sigma_1,\sigma_2 \in C^1_{\mathcal{H}}.$$

It hence follows that a weak material derivative $\partial^\bullet u \in L^2_{V^*}$ of a function $u \in L^2_V$ satisfies the relation

$$\int_0^T \int_\Omega \langle \partial^\bullet u, \varphi \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} = -\int_0^T \int_\Omega (u, \partial^\bullet \varphi)_{\mathcal{H}(t)} - \int_0^T \int_\Omega \tilde{\lambda}(t; u, \varphi) \qquad \forall \varphi \in \mathcal{D}_V(0,T).$$

By choosing the test function $\varphi \in \mathcal{D}_V(0,T) \cong L^2(\Omega; \mathcal{D}_\mathcal{V}(0,T))$ to be of the form $\varphi(t,\omega) = \phi(t)\zeta(\omega)$ for an arbitrary $\phi \in \mathcal{D}_\mathcal{V}(0,T)$ and $\zeta \in L^2(\Omega)$, and interchanging integrals with Fubini's theorem, we deduce that the realisations of the weak material derivative coincide with the weak material derivative of its realisations, i.e.

$$(\partial^\bullet u)(t,\omega) = (\partial^\bullet u_\omega)(t) \quad \text{in } \mathcal{V}^*(t).$$

Thus we obtain the following isomorphisms.

Theorem 3.3.2 (Sobolev-Bochner space isomorphism)**.** *The evolving random Sobolev-Bochner spaces $W(V,V^*)$ and $W(V,H)$ are isometrically isomorphic to the following spaces*

$$W(V,V^*) \cong L^2(\Omega; W(\mathcal{V},\mathcal{V}^*))$$
$$W(V,H) \cong L^2(\Omega; W(\mathcal{V},\mathcal{H})).$$

3.4 Applications of the extended domain mapping method and an abstract analysis of the reformulated problem

We continue by providing an abstract framework in which we treat the general form of the stochastic partial differential equations defined over the deterministically evolving reference domain, which arise after applying the extended domain mapping method to an initial advection-diffusion problem on a randomly evolving curved domain. This framework will be based upon [2], adopting the notation subsequently presented in [35], where we develop upon the applications to our random geometric settings. Note that in the general framework, we have included an advective term which may occur due to the possible disagreement between the prescribed stochastic domain mapping and the flow associated to the random physical velocity field on the evolving domain or may also arise from an artificial advection introduced in an ALE framework. See later examples of applications of the extended domain mapping method for further discussion of how this advective term may arise. We now continue by outlining the general framework presented in [2, 35].

3.4.1 An abstract analysis for the reformulated problem

After employing the extended domain mapping method to reformulate the initial advection-diffusion problem posed over the randomly evolving curved domain onto the evolving reference domain, the general setting for the mean-weak formulation in an abstract notation will be of the following form. We will have an evolving Hilbert triple

$$V(t) \subset H(t) \subset V^*(t)$$

at each $t \in [0, T]$, as in the previous section 3.3.2, that will represent the random function spaces on the time-dependent reference domain, and where prescribed evolution of these spaces

$$\phi_t : H_0 \to H(t) \qquad \phi_t|_{V_0} : V_0 \to V(t)$$

is defined pathwise by the associated flow map of the given deterministic material velocity field v of the reference domain. That is to say, realisations of the strong material derivative $(\partial^\bullet f)(\omega, t)$ are precisely given by $(\partial_v^\bullet f_\omega)(t)$. We will further have the following time-dependent bilinear forms

$$\begin{aligned} m(t; \cdot, \cdot) &: H(t) \times H(t) \to \mathbb{R} \\ a(t; \cdot, \cdot) &: V(t) \times V(t) \to \mathbb{R} \\ b(t; \cdot, \cdot) &: H(t) \times V(t) \to \mathbb{R}, \end{aligned}$$

respectively relating to the pull-back of the mass term, diffusion term, and possible advection term as previously discussed. We assume that these bilinear forms satisfy the following conditions:

Assumption 3.4.1 (Uniform bounds)**.** *We assume that $m(t; \cdot, \cdot)$ and $a(t; \cdot, \cdot)$ are both symmetric*

$$m(t; \varphi, \psi) = m(t; \psi, \varphi) \qquad a(t; \varphi, \psi) = a(t; \psi, \varphi)$$

and furthermore that there exists constants $C_i > 0$ independent of $t \in [0, T]$ such that

$$C_1 \|\varphi\|^2_{H(t)} \leq m(t; \varphi, \varphi) \leq C_2 \|\varphi\|^2_{H(t)} \qquad \forall \varphi \in H(t) \tag{M1}$$

$$|a(t; \varphi, \psi)| \leq C_3 \|\varphi\|_{V(t)} \|\psi\|_{V(t)} \qquad \forall \varphi, \psi \in V(t), \tag{A1}$$

$$a(t; \varphi, \varphi) \geq C_4 \|\varphi\|^2_{V(t)} - C_5 \|\varphi\|^2_{H(t)} \qquad \forall \varphi \in V(t), \tag{A2}$$

$$|b(t; \varphi, \psi)| \leq C_6 \|\varphi\|_{H(t)} \|\psi\|_{V(t)} \qquad \forall \varphi \in H(t), \psi \in V(t). \tag{B1}$$

Assumption 3.4.2 (Transport property)**.** *We further assume that there exists time-dependent bilinear forms*

$$\begin{aligned} g(t, v; \cdot, \cdot) &: H(t) \times H(t) \to \mathbb{R} \\ \tilde{a}(t, v; \cdot, \cdot) &: V(t) \times V(t) \to \mathbb{R} \\ \tilde{b}(t, v; \cdot, \cdot) &: H(t) \times V(t) \to \mathbb{R}, \end{aligned}$$

such that we have the following abstract transport properties

$$\frac{d}{dt}m(t;\varphi,\psi) = m(t;\partial^\bullet\varphi,\psi) + m(t;\varphi,\partial^\bullet\psi) + g(t,v;\varphi,\psi) \quad \forall\varphi,\psi \in C^1_H \tag{T1}$$

$$\frac{d}{dt}a(t;\varphi,\psi) = a(t;\partial^\bullet\varphi,\psi) + a(t;\varphi,\partial^\bullet\psi) + \tilde{a}(t,v;\varphi,\psi) \quad \forall\varphi,\psi \in C^1_V \tag{T2}$$

$$\frac{d}{dt}b(t;\varphi,\psi) = b(t;\partial^\bullet\varphi,\psi) + b(t;\varphi,\partial^\bullet\psi) + \tilde{b}(t,v;\varphi,\psi) \quad \forall\varphi \in C^1_H, \psi \in C^1_V, \tag{T3}$$

where the above derivative exists classically.

Assumption 3.4.3 (Uniform bounds on derivatives)**.** *We finally assume that the above bilinear forms are uniformly bounded in time. More precisely, that there exists constants $C_i > 0$ independent of $t \in [0,T]$ such that we have*

$$|g(t,v;\varphi,\psi)| \leq C_1\|\varphi\|_{H(t)}\|\psi\|_{H(t)} \quad \forall\varphi,\psi \in H(t) \tag{M2}$$

$$|\tilde{a}(t,v;\varphi,\psi)| \leq C_2\|\varphi\|_{V(t)}\|\psi\|_{V(t)} \quad \forall\varphi,\psi \in V(t) \tag{A3}$$

$$|\tilde{b}(t,v;\varphi,\psi)| \leq C_3\|\varphi\|_{H(t)}\|\psi\|_{V(t)} \quad \forall\varphi \in H(t), \psi \in V(t). \tag{B2}$$

Remark 3.4.1. *Note that in the above bilinear forms, v is purely notational to indicate which velocity field is referred to in the transport property. In particular, we will refer to the smooth velocity of the evolving reference domain by v and in our later numerical analysis, V_h, v_h will respectively indicate the discrete and lifted discrete velocity fields.*

The mean-weak formulation for the reformulated equation/system will then be of the following form in the abstract notation. See section 3.3.2, for a review of the abstract evolving random Sobolev-Bochner spaces.

Problem 3.4.1 (Abstract mean-weak formulation)**.** *Given $u_0 \in V_0$, find $u \in W(V,H)$ such that for a.e. $t \in [0,T]$,*

$$m(t;\partial^\bullet u,\varphi) + g(t,v;u,\varphi) + a(t;u,\varphi) + b(t;u,\varphi) = 0 \tag{3.4.1}$$

for all $\varphi \in V(t)$ and furthermore

$$u(0) = u_0 \quad \textit{in } H_0.$$

Note that the initial condition may be understood to be satisfied due to the continuous embedding $W(V,V^*) \hookrightarrow C^0_H$ established in [2, Lemma 2.35]. The existence of a solution to the abstract mean-weak formulation may now be proved following a standard Galerkin argument based on the following energy equations

$$\frac{1}{2}\frac{d}{dt}m(t;u,u) + \frac{1}{2}g(t,v;u,u) + a(t;u,u) + b(t;u,u) = 0$$

and

$$m(t;\partial^\bullet u,\partial^\bullet u)+g(t,v;u,\partial^\bullet u)+\frac{1}{2}\frac{d}{dt}a(t;u,u)+\frac{d}{dt}b(t;u,u)$$
$$=\frac{1}{2}\tilde{a}(t,v;u,u)+\tilde{b}(t,v;u,u)+b(t;\partial^\bullet u,u),$$

and the previous assumptions on the bilinear forms. See [2, Section 5] and [30, Theorem 4.4] for further discussion of well-posedness for continuous problems of this form.

Theorem 3.4.1 (Well-posedness)**.** *Given $u_0\in V_0$, there exists a unique solution to the mean-weak formulation which satisfies the following energy estimates*

$$\sup_{t\in[0,T]}\|u(t)\|^2_{H(t)}+\int_0^T\|u(t)\|^2_{V(t)}\,dt\lesssim\|u_0\|^2_{H_0} \tag{3.4.2}$$

$$\int_0^T\|\partial^\bullet u(t)\|^2_{H(t)}\,dt+\sup_{t\in[0,T]}\|u(t)\|^2_{V(t)}\lesssim\|u_0\|^2_{V_0}. \tag{3.4.3}$$

We now proceed by applying the extended domain mapping method to our two model advection-diffusion equations on randomly evolving domains. In particular, we will first reformulate the equation/system onto the evolving reference domain with the calculations of the pull-back of differential operators provided in section 3.2, and then verify all the listed assumptions in the abstract analysis are indeed satisfied for a given mean-weak formulation of the reformulated equation/system. Note that the subsequent analysis is motivated by [35], in which the deterministic analogue of each problem is considered.

3.4.2 First application of the EDMM to an advection-diffusion equation on a randomly evolving surface

Let $\Gamma_\omega(t), t\in[0,T]$, denote the smooth randomly evolving compact hypersurface described in section 3.1.1, whose points on the surface evolve by the prescribed random material velocity field

$$v^{\Gamma_\omega}=v^{\Gamma_\omega}_\nu+v^{\Gamma_\omega}_\tau.$$

We impose the following conditions on the tangential component of the random velocity field.

Assumption 3.4.4 (Random velocity field)**.** *We assume that the tangential component $v^{\Gamma_\omega}_\tau$ of the random velocity field v^{Γ_ω} on $\Gamma_\omega(t)$ is uniformly bounded as follows*

$$\|v^{\Gamma_\omega}_\tau\|_{C^1(\Gamma_{\omega,T})}\le C\qquad\Gamma_{\omega,T}=\bigcup_{t\in[0,T]}\Gamma_\omega(t)\times\{t\} \tag{3.4.4}$$

for a constant $C>0$ independent of ω.

The advection-diffusion equation on $\Gamma_\omega(t)$ under consideration for the extended domain mapping method is as follows.

Problem 3.4.2 (Advection-diffusion equation on $\Gamma_\omega(t)$)**.** *Given random initial data $u_0(\omega)$: $\Gamma_\omega(0) \to \mathbb{R}$, find for a.e. $\omega \in \Omega$, $u_\omega(t;\cdot) : \Gamma_\omega(t) \to \mathbb{R}$ such that*

$$\begin{aligned} \partial^\bullet_{v^{\Gamma_\omega}} u_\omega + u_\omega \nabla_{\Gamma_\omega(t)} \cdot v^{\Gamma_\omega} - \Delta_{\Gamma_\omega(t)} u_\omega = 0 \qquad & \text{on } \Gamma_\omega(t), \\ u_\omega(0) = u_0(\omega) \quad & \text{on } \Gamma_\omega(0). \end{aligned}$$

We assume the chosen computational deterministic reference domain for the extended domain mapping method is given by a smooth compact evolving hypersurface $\Gamma_0(t)$, whose evolution from its initial configuration $\Gamma_0(0)$, is determined by the normal velocity field $v_\nu^{\Gamma_0}$. We further assume that the prescribed stochastic domain mapping

$$\phi : \Omega \times \mathcal{G}_T^{\Gamma_0} \to \mathbb{R}^{n+1} \qquad \mathcal{G}_T^{\Gamma_0} = \bigcup_{t \in [0,T]} \Gamma_0(t) \times \{t\},$$

defined over the space-time surface $\mathcal{G}_T^{\Gamma_0}$, which characterises the randomly evolving surface, i.e. such that the mapping

$$\phi_\omega(t;\cdot) : \Gamma_0(t) \to \Gamma_\omega(t) \tag{3.4.5}$$

is a diffeomorphism between the surfaces at each $t \in [0,T]$, further satisfies the following assumptions.

Assumption 3.4.5 (Stochastic domain mapping)**.** *We assume the prescribed stochastic mapping satisfies the following conditions.*

1. ***Measurability:*** *The mapping $\phi : \Omega \times \mathcal{G}_T^{\Gamma_0} \to \mathbb{R}^{n+1}$ is $\mathcal{F} \otimes \mathcal{B}(G_T^{\Gamma_0})-$measurable.*

2. ***Regularity:*** *For a.e. ω, we have $\phi_\omega \in C^2(\mathcal{G}_T^{\Gamma_0})$.*

3. ***Uniform bounds:*** *There exists constants $C_i > 0$ independent of ω and t, such that*

$$\|\phi_\omega\|_{C^2(\mathcal{G}_T^{\Gamma_0})} \le C_1 \tag{3.4.6}$$

$$\|\nabla_{\Gamma_\omega(t)} \phi_\omega(t)^{-1}\|_{L^\infty(\Gamma_\omega(t))} \le C_2, \tag{3.4.7}$$

with $\phi_\omega(t)^{-1}$ denoting the inverse of the diffeomorphic mapping (3.4.5).

We may now employ the computations for the pull-back of the Laplace-Beltrami operator Lemma 2.3.3, and the material derivative Corollary 3.2.2, to arrive at the following reformulated equation for the pull-back of the path-wise solution $\hat{u}_\omega := u_\omega \circ \phi_\omega$.

Problem 3.4.3 (Reformulated equation on $\Gamma_0(t)$)**.** *For a.e. $\omega \in \Omega$, find $\hat{u}_\omega(\cdot,t) : \Gamma_0(t) \to \mathbb{R}$, such that*

$$\begin{aligned} &\partial^\circ(\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega) + \sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega \nabla_{\Gamma_0(t)} \cdot v_\nu^{\Gamma_0} \\ &\quad + \nabla_{\Gamma_0(t)} \cdot \left(\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega v_{\tau,corr}^{\Gamma_0}(\omega)\right) - \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega) \nabla_{\Gamma_0(t)} \hat{u}_\omega) = 0 \quad \text{on } \Gamma_0(t), \\ &\hat{u}_\omega(0) = \hat{u}_0(\omega) \quad \text{on } \Gamma_0(0). \end{aligned}$$

Here, the random coefficients which are defined over the space-time surface $\mathcal{G}_T^{\Gamma_0}$, are given by

$$G_{\Gamma_0}(\omega) = \nabla_{\Gamma_0(t)}\phi_\omega^\top \nabla_{\Gamma_0(t)}\phi_\omega + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} \tag{3.4.8}$$

$$g_{\Gamma_0}(\omega) = det(G_{\Gamma_0}(\omega)) \tag{3.4.9}$$

$$v_{\tau,corr}^{\Gamma_0}(\omega) = G_{\Gamma_0}^{-1}(\omega)\nabla_{\Gamma_0(t)}\phi_\omega^\top \left(v_\tau^{\Gamma_\omega} \circ \phi_\omega - \partial^\circ \phi_\omega\right). \tag{3.4.10}$$

Remark 3.4.2 (Corrective advective term)**.** *In the case where the stochastic domain mapping (3.4.5) is selected to coincide with the flow map associated to the random material velocity field v^{Γ_ω}, we have that $v_{\tau,corr}^{\Gamma_0} = 0$ and consequently there is no random advective term in the reformulated equation. However, in certain applications it may be convenient not to limit the particular choice of the parametrisation of the randomly evolving surface. We will therefore instead choose to focus on the general form of the reformulated equation which includes an additional corrective random advection.*

We may interpret the reformulated equation for the pull-back of the path-wise solution, as a random advection-diffusion process posed over an evolving surface $\Gamma_0(t)$ with material velocity $v_\nu^{\Gamma_0}$. A finite element approximation to a variational formulation of the reformulated equation, may then be proposed following the evolving surface finite element method [30], in which piecewise linear finite elements are adopted on an interpolating surface whose nodes $\{X_j(t)\}_{j=1}^N$ evolve by the prescribed material velocity

$$X_j'(t) = v_\nu^{\Gamma_0}(X_j(t), t) \quad j = 1, ..., N.$$

of the smooth surface $\Gamma_0(t)$. However, by choosing to evolve the nodes of the triangulation in this manner may quickly lead to mesh degeneration in many computations, and consequently will require a suitable remeshing process. To advert this issue, we instead follow [39, 40] and consider an arbitrary Lagrangian-Eulerian setting, in which we introduce an arbitrary tangential velocity field $v_{\tau,arb}^{\Gamma_0}$ on $\Gamma_0(t)$, for which we impose the following assumptions.

Assumption 3.4.6 (ALE velocity)**.** *We assume that the flow map $\tilde{G}(\cdot, t) : \Gamma_0(0) \to \Gamma_0(t)$ associated to the velocity field $v^{\Gamma_0} = v_\nu^{\Gamma_0} + v_{\tau,arb}^{\Gamma_0}$, i.e.*

$$\tilde{G}_t(\cdot, t) = v^{\Gamma_0}(\tilde{G}(\cdot, t), t)$$

satisfies $\tilde{G} \in C^2([0, T]; C^2(\Gamma_0(0)))$ and we furthermore assume

$$v^{\Gamma_0}(\cdot, t) \in C^2(\Gamma_0(t)) \quad \forall t \in [0, T]. \tag{3.4.11}$$

The points on the surface $\Gamma_0(t)$ may now instead be considered to evolve by the material velocity field

$$v^{\Gamma_0} = v_\nu^{\Gamma_0} + v_{\tau,arb}^{\Gamma_0}$$

and an equivalent advection-diffusion equation to the reformulated equation with the new material velocity field may be considered with the following identity

$$\partial^\circ f + f\nabla_{\Gamma_0(t)} \cdot v_\nu^{\Gamma_0} = \partial^\bullet_{v^{\Gamma_0}} f + f\nabla_{\Gamma_0(t)} \cdot v^{\Gamma_0} - \nabla_{\Gamma_0(t)} \cdot (f v^{\Gamma_0}_{\tau,arb}).$$

The result is the following problem.

Problem 3.4.4 (ALE formulation on $\Gamma_0(t)$)**.** *For a.e. $\omega \in \Omega$, find $\hat{u}_\omega(\cdot,t) : \Gamma_0(t) \to \mathbb{R}$, such that*

$$\begin{aligned}
&\partial^\bullet_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega) + \sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega \nabla_{\Gamma_0(t)} \cdot v^{\Gamma_0} \\
&+ \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega(v^{\Gamma_0}_{\tau,corr}(\omega) - v^{\Gamma_0}_{\tau,arb})) - \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)} G^{-1}_{\Gamma_0}(\omega)\nabla_{\Gamma_0(t)}\hat{u}_\omega) = 0 \quad \text{on } \Gamma_0(t), \\
&\hat{u}_\omega(0) = \hat{u}_0(\omega) \quad \text{on } \Gamma_0(0).
\end{aligned}$$

Remark 3.4.3 (Constructing an ALE velocity field)**.** *A general method to generate a tangential velocity field which preserves a good mesh quality for an arbitrary evolving surface is beyond the scope of this thesis. Instead, we refer the reader to [36, 68] in which a reparametrisation based upon solutions to the harmonic heat flow map on manifolds is considered, and more recently [60] in which the ALE velocity is constructed from a spring system, based on the connectivity of the nodes in the mesh.*

We continue by deriving a mean-weak formulation for the random advection-diffusion equation on the evolving reference domain, Problem 3.4.4, and verify that all the listed assumptions in the abstract analysis are indeed satisfied, guaranteeing the existence and uniqueness of a solution. In the notation of the abstract framework, the random function spaces are given by

$$\begin{aligned}
V(t) &= L^2(\Omega; H^1(\Gamma_0(t))) \\
H(t) &= L^2(\Omega; L^2(\Gamma_0(t))) \\
V^*(t) &= L^2(\Omega; H^{-1}(\Gamma_0(t))),
\end{aligned}$$

and the push-forward operator

$$\phi_t : L^2(\Gamma_0(0)) \to L^2(\Gamma_0(t))$$

is given by the associated flow map of the ALE velocity v^{Γ_0}. More precisely, the push-forward of a function $u_0 \in L^2(\Gamma_0(0))$ onto $L^2(\Gamma_0(t))$ is defined by

$$\phi_t u_0(\tilde{G}(x,t),t) = u_0(x,t) \quad x \in \Gamma_0(0).$$

By the assumed regularity of the mapping $\tilde{G}$ given in Assumption 3.4.6, it follows that $(L^2(\Gamma_0), \phi_t)$ and $(H^1(\Gamma_0), \phi_t|_{L^2(\Gamma_0(0))})$ are both compatible pairs, as proved in A.1.2. We therefore deduce that (H, ϕ_t) and $(V, \phi_t|_{V(0)})$ are also compatible by Lemma 3.3.1. The abstract time-dependent bilinear forms in the case of Problem 3.4.4 are given by

$$\begin{aligned}
m(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}\varphi\psi \\
a(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0(t)}\varphi \cdot \nabla_{\Gamma_0(t)}\psi \\
b(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}\varphi \nabla_{\Gamma_0(t)}\psi \cdot (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0}) \\
g(t,v^{\Gamma_0};\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \left(\partial^\bullet_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}} \nabla_{\Gamma_0(t)} \cdot v^{\Gamma_0}\right) \varphi\psi,
\end{aligned}$$

where here we have intergrated by parts with the identity

$$\int_{\Gamma_0(t)} \nabla_{\Gamma_0(t)} \cdot f = \int_{\Gamma_0(t)} f \cdot H^{\Gamma_0}\nu^{\Gamma_0} + \int_{\partial\Gamma_0(t)} f \cdot \mu^{\Gamma_0}$$

noting that $\partial\Gamma_0(t) = \emptyset$ as $\Gamma_0(t)$ is assumed to be compact. The mean-weak formulation for Problem 3.4.4 will now be of form described in the abstract setting which we shall restate below for convenience. As we will henceforth only be concerned with the formulation on the reference domain, we will we drop the pull-back notation $\hat{u}$ for the pathwise solution and instead write u.

Problem 3.4.5 (Mean-weak formulation). *Given $u_0 \in V_0$, find $u \in W(V,H)$ such that for a.e. $t \in [0,T]$*

$$m(t;\partial^\bullet u,\varphi) + g(t,v^{\Gamma_0};u,\varphi) + a(t;u,\varphi) + b(t;u,\varphi) = 0 \tag{3.4.12}$$

for all $\varphi \in V(t)$, and which furthermore satisfies the initial condition

$$u(0) = u_0 \quad \text{in } H_0.$$

In order to verify that all the assumptions listed in the abstract analysis are indeed satisfied, we will first begin by deriving uniform bounds on the random coefficients based upon the assumed estimates on the stochastic domain mapping and the random tangential velocity field.

Lemma 3.4.1 (Uniform estimates on coefficients). *There exists constant $C_i > 0$ independent of $\omega \in \Omega$ and $t \in [0,T]$ such that*

$$C_1 \le \sqrt{g_{\Gamma_0}(\omega,t;x)} \le C_2 \qquad \forall x \in \Gamma_0(t) \tag{3.4.13}$$

$$C_3|\eta|^2 \le G_{\Gamma_0}(\omega,t;x)\eta \cdot \eta \le C_4|\eta|^2 \qquad \forall x \in \Gamma_0(t), \forall \eta \in T\Gamma_0(t) \tag{3.4.14}$$

$$|v_{\tau,corr}^{\Gamma_0}(\omega,t;x)| \le C_5 \qquad \forall x \in \Gamma_0(t). \tag{3.4.15}$$

Furthermore, such that for all $x \in \Gamma_0(t)$

$$|\partial^\bullet_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}})(\omega, t; x)| \le C_6 \tag{3.4.16}$$

$$|\partial^\bullet_{v^{\Gamma_0}}(G^{-1}_{\Gamma_0})(\omega, t; x)| \le C_7 \tag{3.4.17}$$

$$|\partial^\bullet_{v^{\Gamma_0}}(v^{\Gamma_0}_{\tau,corr})(\omega, t; x)| \le C_8. \tag{3.4.18}$$

Proof. We begin by observing that the tensor $G_{\Gamma_0} = \nabla_{\Gamma_0(t)}\phi_\omega^\top \nabla_{\Gamma_0(t)}\phi_\omega + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0}$ may be decomposed into the following product

$$G_{\Gamma_0}(\omega) = \left(\nabla_{\Gamma_0(t)}\phi_\omega + \nu^{\Gamma_\omega} \circ \phi_\omega \otimes \nu^{\Gamma_0}\right)^\top \left(\nabla_{\Gamma_0(t)}\phi_\omega + \nu^{\Gamma_\omega} \circ \phi_\omega \otimes \nu^{\Gamma_0}\right).$$

by the orthoganality of $\nabla_{\Gamma_0(t)}\phi_\omega(\nu^{\Gamma_\omega} \circ \phi_\omega) = 0$. We may then easily invert each term, leading to the following expression for the inverse of G_{Γ_0},

$$\begin{aligned} G^{-1}_{\Gamma_0}(\omega) &= \left(\nabla_{\Gamma_\omega(t)}\phi_\omega(t)^{-1} + \nu^{\Gamma_0} \otimes \nu^{\Gamma_\omega} \circ \phi_\omega\right)^\top \left(\nabla_{\Gamma_\omega(t)}\phi_\omega(t)^{-1} + \nu^{\Gamma_0} \otimes \nu^{\Gamma_\omega} \circ \phi_\omega\right)^\top \\ &= \nabla_{\Gamma_\omega(t)}\phi_\omega(t)^{-\top}\nabla_{\Gamma_\omega(t)}\phi_\omega(t)^{-1} + \nu^{\Gamma_\omega} \circ \phi_\omega \otimes \nu^{\Gamma_\omega} \circ \phi_\omega. \end{aligned}$$

The uniform estimates (3.4.13) and (3.4.14) now follow as a result of the uniform bounds on the stochastic domain mapping (3.4.6) and its inverse (3.4.7). Similarly, the uniform bound on the corrective advection term

$$v^{\Gamma_0}_{\tau,corr}(\omega) = G^{-1}_{\Gamma_0}(\omega)\nabla_{\Gamma_0(t)}\phi_\omega^\top \left(v^{\Gamma_\omega}_\tau \circ \phi_\omega - \partial^\circ \phi_\omega\right)$$

will also follow from the estimates on the stochastic mapping and the assumed uniform bound (3.4.4) on the random tangential velocity of the surface $\Gamma_\omega(t)$. For the estimates (3.4.16-3.4.18) on the material derivative of the random coefficients we observe

$$\begin{aligned} \partial^\bullet_{v^{\Gamma_0}}\left(\sqrt{g_{\Gamma_0}(\omega)}\right) &= \frac{\sqrt{g_{\Gamma_0}(\omega)}}{2}\, trace\left(G^{-1}_{\Gamma_0}(\omega)\partial^\bullet_{v^{\Gamma_0}} G_{\Gamma_0}(\omega)\right) \\ \partial^\bullet_{v^{\Gamma_0}} G^{-1}_{\Gamma_0}(\omega) &= -G^{-1}_{\Gamma_0}(\omega)\partial^\bullet_{v^{\Gamma_0}} G_{\Gamma_0}(\omega) G^{-1}_{\Gamma_0}(\omega). \end{aligned}$$

Hence with the identity

$$\partial^\bullet_{v^{\Gamma_0}}\nu^{\Gamma_0} = -(\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top \nu^{\Gamma_0},$$

which may be quickly derived by expressing the unit normal as $\nu^{\Gamma_0} = \nabla d^{\Gamma_0}$ and interchanging derivatives, we compute

$$\begin{aligned} \partial^\bullet_{v^{\Gamma_0}} G_{\Gamma_0}(\omega) = \partial^\bullet_{v^{\Gamma_0}} &\nabla_{\Gamma_0(t)}\phi_\omega^\top \nabla_{\Gamma_0(t)}\phi_\omega + \nabla_{\Gamma_0(t)}\phi_\omega^\top \partial^\bullet_{v^{\Gamma_0}} \nabla_{\Gamma_0(t)}\phi_\omega \\ &- (\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} - \nu^{\Gamma_0} \otimes (\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top \nu^{\Gamma_0} \end{aligned}$$

and deduce the uniform bounds (3.4.16) and (3.4.17) on $\partial^\bullet_{v^{\Gamma_0}}\sqrt{g_{\Gamma_0}}$ and $\partial^\bullet_{v^{\Gamma_0}} G^{-1}$. The uniform bound on the material derivative of the random corrective advection may now be deduced from the bounds (3.4.17) on $\partial^\bullet_{v^{\Gamma_0}} G^{-1}$, the uniform estimates on the stochastic domain mapping

(3.4.6) and the uniform bound on the random tangential velocity field (3.4.4). □

As a consequence of the estimates (3.4.13 - 3.4.15) on the random coefficients, we deduce that the above time-dependent bilinear forms satisfy the assumptions (M1), (A1), (A2) and (B1) given in the abstract analysis. Specifically, that

$$\begin{aligned}
C_1\|\varphi\|^2_{H(t)} \leq m(t;\varphi,\varphi) \leq C_2\|\varphi\|^2_{H(t)} \qquad & \forall \varphi \in H(t) \\
|a(t;\varphi,\psi)| \leq C_3\|\varphi\|_{V(t)}\|\psi\|_{V(t)} \qquad & \forall \varphi,\psi \in V(t), \\
a(t;\varphi,\varphi) \geq C_4\|\varphi\|^2_{V(t)} - C_5\|\varphi\|^2_{H(t)} \qquad & \forall \varphi \in V(t), \\
|b(t;\varphi,\psi)| \leq C_6\|\varphi\|_{H(t)}\|\psi\|_{V(t)} \qquad & \forall \varphi \in H(t), \psi \in V(t).
\end{aligned}$$

We next show that the abstract transport properties (T1, T2, T3) hold. For this, we first note that since (H,ϕ_t) is a compatible pair, it follows from Theorem A.1.1 that for any $\varphi,\psi \in C^1_H$, the mapping

$$t \mapsto (\varphi(t),\psi(t))_{H(t)}$$

is absolutely continuous on $[0,T]$. Furthermore, the derivative is given a.e. by

$$\frac{d}{dt}(\varphi,\psi)_{H(t)} = \int_\Omega \int_{\Gamma_0(t)} \partial^\bullet_{v^{\Gamma_0}}\varphi\,\psi + \varphi\,\partial^\bullet_{v^{\Gamma_0}}\psi + \nabla_{\Gamma_0(t)}\cdot v^{\Gamma_0}\,\varphi\psi, \tag{3.4.19}$$

where the extra term in the transport property in the case of random function space $H(t)$ has been identified in equation (3.3.12) as precisely the expectation of the additional extra term arising in the equivalent deterministic transport property

$$t \mapsto (\sigma_1,\sigma_2)_{L(\Gamma_0(t))} \qquad \sigma_1,\sigma_2 \in L^2(\Gamma_0(t))$$

which has previously been calculated [42, Theorem 5.1], giving (3.4.19). Differentiating the bilinear forms

$$\begin{aligned}
m(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}\varphi\,\psi \\
a(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0}\nabla_{\Gamma_0(t)}\varphi\cdot\nabla_{\Gamma_0(t)}\psi \\
b(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}\varphi\nabla_{\Gamma_0(t)}\psi\cdot
\end{aligned}$$

with the identity

$$\partial^\bullet_{v^{\Gamma_0}}(\nabla_{\Gamma_0(t)}f) = \nabla_{\Gamma_0(t)}(\partial^\bullet_{v^{\Gamma_0}}f) - (\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top\nabla_{\Gamma_0(t)}f + (\nu^{\Gamma_0}\otimes(\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top\nu^{\Gamma_0})\nabla_{\Gamma_0(t)}f.$$

derived in [32, Lemma 2.6], we obtain the following transport properties

$$
\begin{aligned}
\frac{d}{dt}m(t;\varphi,\psi) &= m(t;\partial^{\bullet}_{v^{\Gamma_0}}\varphi,\psi) + m(t;\varphi,\partial^{\bullet}_{v^{\Gamma_0}}\psi) + g(t,v^{\Gamma_0};\varphi,\psi)\\
\frac{d}{dt}a(t;\varphi,\psi) &= a(t;\partial^{\bullet}_{v^{\Gamma_0}}\varphi,\psi) + a(t;\varphi,\partial^{\bullet}_{v^{\Gamma_0}}\psi) + \tilde{a}(t,v^{\Gamma_0};\varphi,\psi)\\
\frac{d}{dt}b(t;\varphi,\psi) &= b(t;\partial^{\bullet}_{v^{\Gamma_0}}\varphi,\psi) + b(t;\varphi,\partial^{\bullet}_{v^{\Gamma_0}}\psi) + \tilde{b}(t,v^{\Gamma_0};\varphi,\psi)
\end{aligned}
$$

where the additional bilinear forms are identified as

$$\tilde{a}(t,v^{\Gamma_0};\varphi,\psi) = \int_\Omega\int_{\Gamma_0(t)} \tilde{\mathcal{A}}(v^{\Gamma_0})\nabla_{\Gamma_0(t)}\varphi\cdot\nabla_{\Gamma_0(t)}\psi \tag{3.4.20}$$

$$\tilde{b}(t,v^{\Gamma_0};\varphi,\psi) = \int_\Omega\int_{\Gamma_0(t)} \varphi\tilde{\mathcal{B}}(v^{\Gamma_0})\cdot\nabla_{\Gamma_0(t)}\psi, \tag{3.4.21}$$

with random coefficients are given by

$$\tilde{\mathcal{A}}(v^{\Gamma_0}) = \partial^{\bullet}_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}})G^{-1}_{\Gamma_0} + \sqrt{g_{\Gamma_0}}\partial^{\bullet}_{v^{\Gamma_0}}(G^{-1}_{\Gamma_0}) + (\nabla_{\Gamma_0(t)}\cdot v^{\Gamma_0})\sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0} - 2\sqrt{g_{\Gamma_0}}\,\mathcal{D}^{\Gamma_0}(v^{\Gamma_0})$$

and

$$
\begin{aligned}
\tilde{\mathcal{B}}(v^{\Gamma_0}) = &\left(\partial^{\bullet}_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)}\cdot v^{\Gamma_0} - \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)}v^{\Gamma_0}\right)(v^{\Gamma_0}_{\tau,arb} - v^{\Gamma_0}_{\tau,corr})\\
&+ \sqrt{g_{\Gamma_0}}(\partial^{\bullet}_{v^{\Gamma_0}}v^{\Gamma_0}_{\tau,arb} - \partial^{\bullet}_{v^{\Gamma_0}}v^{\Gamma_0}_{\tau,corr}),
\end{aligned}
$$

where the deformation tensor $\mathcal{D}^{\Gamma_0}(v^{\Gamma_0})$ is defined by

$$\mathcal{D}^{\Gamma_0}(v^{\Gamma_0}) = \frac{1}{2}(G^{-1}_{\Gamma_0}(\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top + \nabla_{\Gamma_0(t)}v^{\Gamma_0}G^{-1}_{\Gamma_0}). \tag{3.4.22}$$

Here we note that in our calculations, the terms of the form

$$(\nu^{\Gamma_0}\otimes(\nabla_{\Gamma_0(t)}v^{\Gamma_0})^\top\nu^{\Gamma_0})\nabla_{\Gamma_0(t)}\varphi\cdot\nabla_{\Gamma_0(t)}\psi$$

will vanish with the tensor formula

$$(a\otimes b)c\cdot d = (a\cdot d)(b\cdot c)$$

and the orthogonality of $\nu^{\Gamma_0}\cdot\nabla_{\Gamma_0(t)}\varphi = 0$. It therefore follows from previously established uniform bounds on the material derivative of the random coefficients (3.4.16) (3.4.17) (3.4.18) given in Lemma 3.4.1, and that the random coefficients are uniformly bounded

$$\|\tilde{\mathcal{A}}(v^{\Gamma_0})(\omega,t;x)\|_{\mathbb{R}^{(n+1)\times(n+1)}} \leq C_1 \quad \forall x \in \Gamma_0(t) \tag{3.4.23}$$

$$\|\tilde{\mathcal{B}}(v^{\Gamma_0})(\omega,t;x)\|_{\mathbb{R}^{n+1}} \leq C_2 \quad \forall x \in \Gamma_0(t) \tag{3.4.24}$$

for constants $C_1, C_2 > 0$ independent of ω and t, which thus leads to the uniform boundedness of the bilinear forms $\tilde{a}(t, v^{\Gamma}, \cdot, \cdot)$ and $\tilde{b}(t; v^{\Gamma_0}, \cdot, \cdot)$ giving the assumptions (G2) and (A3). Uniform boundedness (G2) of the bilinear form $g(t, v^{\Gamma_0}; \cdot, \cdot)$ also holds due estimates on $\sqrt{g_{\Gamma_0}}$. Thus all the assumptions listed in the abstract analysis are satisfied giving the following well-posedness result for the mean-weak formulation of Problem 3.4.4.

Theorem 3.4.2 (Well-posedness)**.** *Given $u_0 \in V_0$, there exists a unique solution $u \in W(V,H)$ to the mean-weak formulation which furthermore satisfies the following energy bounds*

$$\sup_{t\in[0,T]} \|u(t)\|^2_{H(t)} + \int_0^T \|u(t)\|^2_{V(t)}\, dt \lesssim \|u_0\|^2_{H_0} \tag{3.4.25}$$

$$\int_0^T \|\partial^{\bullet}_{v^{\Gamma_0}} u(t)\|^2_{H(t)}\, dt + \sup_{t\in[0,T]} \|u(t)\|^2_{V(t)} \lesssim \|u_0\|^2_{V_0}. \tag{3.4.26}$$

3.4.3 A second application of the EDMM to a coupled advection-diffusion system on a randomly evolving bulk-surface

We proceed in a similar manner to the previous example and apply the extended domain mapping method to our second model problem of a coupled advection-diffusion system on a randomly evolving bulk-surface. The consideration of this model problem was motivated by [35], in which the deterministic analogue is analysed. We denote the randomly evolving compact smooth surface as described in section 3.1.1 by $\Gamma_\omega(t) \subset \mathbb{R}^{n+1}, t \in [0,T]$ and its interior bulk domain by $D_\omega(t)$. We will assume that realisations of the given random physical material velocity field

$$w_{phys,\omega}(t;\cdot) : \overline{D_\omega(t)} \to \mathbb{R}^{n+1} \tag{3.4.27}$$

are continuous over the bulk surface $\overline{D_\omega(t)}$. In other words, we will assume that the physical process driving the advection of material points within the bulk is also the same process as is found on the surface. Note that this assumption is only for convenenice and the subsequent

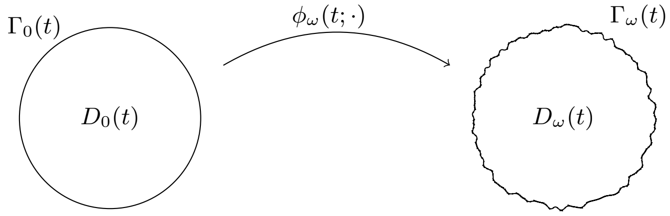

Figure 3.4: A cross-section of the evolving computational bulk-surface reference domain and a realisation of the randomly evolving bulk-surface.

analysis may be easily modified to the consideration of differing bulk and surface random advection processes. We denote the decomposition of the velocity field $w_{phys,\omega}$ at the random surface boundary, into its normal and tangential component by

$$w_{phys,\omega} = w_{\nu}^{\Gamma_\omega} + w_{\tau}^{\Gamma_\omega},$$

and impose the following assumptions.

Assumption 3.4.7 (Random physical velocity field)**.** *We assume the random physical velocity field w_{phys} of the randomly evolving bulk-surface $\overline{D_\omega(t)}$ is uniformly bounded as follows*

$$\|w_{phys,\omega}\|_{C^1(D_{\omega,T})} \leq C_1 \qquad D_{\omega,T} = \bigcup_{t\in[0,T]} D_\omega(t) \times \{t\} \tag{3.4.28}$$

$$\|w_{\tau,\omega}^{\Gamma_\omega}\|_{C^1(\Gamma_{\omega,T})} \leq C_2 \qquad \Gamma_{\omega,T} = \bigcup_{t\in[0,T]} \Gamma_\omega(t) \times \{t\} \tag{3.4.29}$$

for constants $C_i > 0$ independent of ω.

The model coupled advection-diffusion system on the randomly evolving bulk-surface under consideration for the extended domain mapping method reads as follows.

Problem 3.4.6 (Coupled system on random bulk-surface)**.** *Given the random initial data $u_0(\omega) : \overline{D_\omega(0)} \to \mathbb{R}$ and $v_0(\omega) : \Gamma_\omega(0) \to \mathbb{R}$, find for a.e. $\omega \in \Omega$, a pair $(u_\omega(\cdot,t), v_\omega(\cdot,t)) : \overline{D_\omega(t)} \times \Gamma_\omega(t) \to \mathbb{R}^2$ such that*

$$\partial^\bullet_{w_{phys}} u_\omega + u_\omega \nabla \cdot w_{phys,\omega} - \Delta u_\omega = 0 \qquad \text{in } D_\omega(t) \tag{3.4.30}$$

$$\alpha u_\omega - \beta v_\omega + \frac{\partial u_\omega}{\partial \nu^{\Gamma_\omega}} = 0 \qquad \text{on } \Gamma_\omega(t) \tag{3.4.31}$$

$$\partial^\bullet_{w_{phys}} v_\omega + v_\omega \nabla_{\Gamma_\omega(t)} \cdot w_{phys,\omega} - \Delta_{\Gamma_\omega(t)} v_\omega + \frac{\partial u_\omega}{\partial \nu^{\Gamma_\omega}} = 0 \qquad \text{on } \Gamma_\omega(t). \tag{3.4.32}$$

Here $\alpha, \beta > 0$ are positive given constants. The reference domain for the extended domain mapping method is selected to be an evolving compact smooth hypersurface $\Gamma_0(t)$, whose evolution from its initial configuation $\Gamma_0(0)$ is described by the normal velocity field

$$w_\nu^{\Gamma_0}(\cdot,t) : \Gamma_0(t) \to \mathbb{R}^{n+1}$$

and which bounds an evolving interior bulk domain $D_0(t)$. The prescribed stochastic domain mapping defined over the evolving bulk-surface reference domain

$$\phi : \Omega \times \mathcal{G}_T^{\overline{D_0}} \to \mathbb{R}^{n+1} \qquad \mathcal{G}_T^{\overline{D_0}} := \bigcup_{t\in[0,T]} \overline{D_0(t)} \times \{t\}, \tag{3.4.33}$$

which characterises the randomly evolving bulk-surface, in the sense the mappings

$$\phi_\omega(t;\cdot)|_{D_0(t)} : D_0(t) \to D_\omega(t) \tag{3.4.34}$$
$$\phi_\omega(t;\cdot)|_{\Gamma_0(t)} : \Gamma_0(t) \to \Gamma_\omega(t) \tag{3.4.35}$$

are diffeomorphisms between the respective reference bulk/surface and the random bulk/surface, will be assumed to satisfy the following conditions.

Assumption 3.4.8 (Stochastic domain mapping). *We impose the following conditions on the stochastic domain mapping.*

1. ***Measurability:*** *The restriction of the stochastic domain mapping (3.4.33) onto the following space-time domains*

$$\mathcal{G}_T^{\Gamma_0} = \bigcup_{t\in[0,T]} \Gamma_0(t) \times \{t\} \qquad \mathcal{G}_T^{D_0} = \bigcup_{t\in[0,T]} D_0(t) \times \{t\},$$

   is respectively $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T^{\Gamma_0})$ and $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T^{D_0})$-measurable.

2. ***Regularity:*** *For a.e. $\omega \in \Omega$, we have*

$$\phi_\omega \in C^0(\mathcal{G}_T^{\overline{D_0}}) \qquad \phi_\omega|_{\mathcal{G}_T^{\Gamma_0}} \in C^2(\mathcal{G}_T^{\Gamma_0}) \qquad \phi_\omega|_{\mathcal{G}_T^{D_0}} \in C^2(\mathcal{G}_T^{D_0}). \tag{3.4.36}$$

3. ***Uniform bounds:*** *There exists constants $C_i > 0$ independent of ω and t such that*

$$\|\phi_\omega\|_{C^2\left(\mathcal{G}_T^{\Gamma_0}\right)} \le C_1 \tag{3.4.37}$$
$$\|\phi_\omega\|_{C^2\left(\mathcal{G}_T^{D_0}\right)} \le C_2 \tag{3.4.38}$$
$$\|\nabla\phi_\omega(t)^{-1}\|_{L^\infty(D_\omega(t))} \le C_3 \tag{3.4.39}$$
$$\|\nabla_{\Gamma_\omega(t)}\phi_\omega(t)^{-1}\|_{L^\infty(\Gamma_\omega(t))} \le C_4, \tag{3.4.40}$$

   with $\phi_\omega(t)^{-1}$ representing the inverse of the diffeomorphic domain mappings (3.4.34) and (3.4.35).

Reformulating the coupled system onto the evolving reference bulk-surface leads to the following random advection-diffusion system for the pull-back of the pathwise bulk solution $\hat{u}_\omega = u_\omega \circ \phi_\omega$ and the surface solution $\hat{v}_\omega = v_\omega \circ \phi_\omega$.

Problem 3.4.7 (Reformulated system). *For a.e. $\omega \in \Omega$, find a pair $(\hat{u}_\omega(\cdot,t), \hat{v}_\omega(\cdot,t)) : \overline{D_0(t)} \times$*

$\Gamma_0(t) \to \mathbb{R}^2$ *such that*

$$\partial_t(\sqrt{g(\omega)}\hat{u}_\omega) + \nabla \cdot (\sqrt{g(\omega)}\hat{u}_\omega w_{corr}(\omega)) - \nabla \cdot (\sqrt{g(\omega)}G^{-1}(\omega)\nabla \hat{u}_\omega) = 0 \quad \text{in } D_0(t) \tag{3.4.41}$$

$$\alpha \hat{u}_\omega - \beta \hat{v}_\omega + \frac{\sqrt{g(\omega)}}{\sqrt{g_{\Gamma_0}(\omega)}} G^{-1}(\omega)\nu^{\Gamma_0} \cdot \nabla \hat{u}_\omega = 0 \quad \text{on } \Gamma_0(t) \tag{3.4.42}$$

$$\begin{aligned}\partial^\circ(\sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega) &+ \sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega \nabla_{\Gamma_0(t)} \cdot w_\nu^{\Gamma_0} + \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega w_{\tau,corr}^{\Gamma_0}(\omega)) \\ &- \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega)\nabla_{\Gamma_0(t)}\hat{v}_\omega) - \sqrt{g_{\Gamma_0}(\omega)}(\alpha \hat{u}_\omega - \beta \hat{v}_\omega) = 0 \quad \text{on } \Gamma_0(t).\end{aligned} \tag{3.4.43}$$

Here the random bulk and surface coefficients are given by

$$G(\omega) = \nabla \phi_\omega^\top \nabla \phi_\omega \qquad G_{\Gamma_0}(\omega) = \nabla_{\Gamma_0}\phi_\omega^\top \nabla_{\Gamma_0}\phi_\omega + \nu^{\Gamma_0} \otimes \nu^{\Gamma_0} \tag{3.4.44}$$

$$g(\omega) = det(G(\omega)) \qquad g_{\Gamma_0}(\omega) = det(G_{\Gamma_0}(\omega)) \tag{3.4.45}$$

and the corrective random velocity fields are given by

$$w_{corr}(\omega) = G^{-1}(\omega)\nabla \phi_\omega^\top (w_{phys,\omega} \circ \phi_\omega - \partial_t \phi_\omega) \qquad \text{in } D_0(t) \tag{3.4.46}$$

$$w_{\tau,corr}^{\Gamma_0}(\omega) = G_{\Gamma_0}^{-1}(\omega)\nabla_{\Gamma_0}\phi_\omega^\top (w_{phys,\omega} \circ \phi_\omega - \partial^\circ \phi_\omega) \quad \text{on } \Gamma_0(t). \tag{3.4.47}$$

Remark 3.4.4 (Corrective velocity fields)**.** *It is worthwhile recalling that if we pull-back trajectories $\{y(s)\}_{s\in[0,T]}$ on the random bulk-surface which evolve by the given random physical velocity field w_{phys}, onto the reference bulk-surface by the given stochastic domain mapping, i.e.*

$$\phi_\omega(x(t), t) = y(t), \qquad y'(t) = v^{\Gamma_\omega}(y(t), t),$$

then the corrective random velocities fields are precisely given by

$$x'(t) = \begin{cases} w_\nu^{\Gamma_0} + w_{\tau,corr}^{\Gamma_0}(\omega) & \text{on } \Gamma_0(t) \\ w_{corr}(\omega) & \text{in } D_0(t). \end{cases}$$

Additionally, we observe that the corrective random bulk velocity $w_{corr}(\omega)$ at the boundary $\Gamma_0(t)$ is given by

$$w_{corr}(\omega) = G^{-1}(\omega)\nabla \phi_\omega^\top (w_{phys,\omega} \circ \phi_\omega - \partial^\circ \phi_\omega + \nabla \phi_\omega w_\nu^{\Gamma_0}) \tag{3.4.48}$$

$$= G^{-1}(\omega)\nabla \phi_\omega^\top (w_{phys,\omega} \circ \phi_\omega - \partial^\circ \phi_\omega) + w_\nu^{\Gamma_0} \tag{3.4.49}$$

$$= G_{\Gamma_0}^{-1}(\omega)\nabla_{\Gamma_0}\phi_\omega^\top (w_{phys,\omega} \circ \phi_\omega - \partial^\circ \phi_\omega) + w_\nu^{\Gamma_0} \tag{3.4.50}$$

$$= w_{\tau,corr}^{\Gamma_0}(\omega) + w_\nu^{\Gamma_0}. \tag{3.4.51}$$

Here we used the assumed compatibility of the stochastic domain mapping,

$$(w_{phys,\omega} \circ \phi_\omega - \partial^\circ \phi_\omega) \cdot \nu^{\Gamma_\omega} \circ \phi_\omega = 0,$$

to deduce

$$G^{-1}(\omega)\nabla\phi_\omega^\top(\mathcal{P}_{\Gamma_\omega}\circ\phi_\omega) = G_{\Gamma_0}^{-1}\nabla_{\Gamma_0}\phi_\omega^\top.$$

We next observe that the above reformulated coupled equations may be considered as a system of random advection-diffusion equations on an evolving bulk-surface domain which evolves by the material velocity field

$$\tilde{w}_{ref} = \begin{cases} 0 & \text{in } D_0(t), \\ w_\nu^{\Gamma_0} & \text{on } \Gamma_0(t). \end{cases} \tag{3.4.52}$$

Continuing our initial discussion of the benefits of considering an ALE setting in a spatial discretisation, as suppose to a Lagrangian formulation based on the given velocity field $\tilde{w}_{ref}$ of the reference domain, we introduce an arbitrary velocity field in the bulk-surface. More precisely, we introduce a velocity field

$$w_{arb}(\cdot,t) : \overline{D_0(t)} \to \mathbb{R}^{n+1} \tag{3.4.53}$$

on the evolving bulk-surface $\overline{D_0(t)}$ which only has a tangential component on the boundary

$$w_{arb}(\cdot,t)\cdot\nu^{\Gamma_0}(\cdot,t) = 0 \quad \text{on } \Gamma_0(t).$$

We denote the tangential component on the surface boundary by $w_{\tau,arb}^{\Gamma_0}$ and define a new material velocity field on the bulk-surface $\overline{D_0(t)}$ by

$$w = \tilde{w}_{ref} + w_{arb} = \begin{cases} w_{arb} & \text{in } D_0(t), \\ w_\nu^{\Gamma_0} + w_{\tau,arb}^{\Gamma_0} & \text{on } \Gamma_0(t). \end{cases}$$

Assumption 3.4.9 (ALE velocity)**.** *We assume that the flow map $\tilde{G}(\cdot,t) : \overline{D_0(0)} \to \overline{D_0(t)}$ associated to the velocity field w, i.e.*

$$\tilde{G}_t(x,t) = w(\tilde{G}(x,t),t) \qquad \forall x \in \overline{D_0(0)}$$

satisfies $\tilde{G} \in C^2([0,T]; C^2(\overline{D_0(0)}))$ and furthermore that we have for all $t \in [0,T]$,

$$w(\cdot,t) \in C^2(D_0(t)) \tag{3.4.54}$$

$$w(\cdot,t) \in C^2(\Gamma_0(t)). \tag{3.4.55}$$

If we now consider points in the evolving bulk-surface $\overline{D_0(t)}$ to evolve by the material velocity field w, then an equivalent advection-diffusion system to the coupled problem 3.4.7, may be written as follows.

Problem 3.4.8 (ALE formulation)**.** *For a.e. $\omega \in \Omega$, find a pair $(\hat{u}_\omega(\cdot,t), \hat{v}_\omega(\cdot,t)) : \overline{D_0(t)} \times$*

$\Gamma_0(t) \to \mathbb{R}^2$ *such that*

$$\partial^\bullet_w(\sqrt{g(\omega)}\hat{u}_\omega) + \sqrt{g(\omega)}\hat{u}_\omega \nabla \cdot w \\ + \nabla \cdot (\sqrt{g(\omega)}\hat{u}_\omega(w_{corr}(\omega) - w)) - \nabla \cdot (\sqrt{g(\omega)}G^{-1}(\omega)\nabla \hat{u}_\omega) = 0 \quad \text{in } D_0(t) \tag{3.4.56}$$

$$\alpha \hat{u}_\omega - \beta \hat{v}_\omega + \frac{\sqrt{g(\omega)}}{\sqrt{g_{\Gamma_0}(\omega)}} G^{-1}(\omega)\nu^{\Gamma_0} \cdot \nabla \hat{u}_\omega = 0 \quad \text{on } \Gamma_0(t) \tag{3.4.57}$$

$$\partial^\bullet_w(\sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega) + \sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega \nabla_{\Gamma_0(t)} \cdot w + \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)}\hat{v}_\omega(w^{\Gamma_0}_{corr}(\omega) - w^{\Gamma_0}_{\tau,arb})) \\ - \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)}G^{-1}_{\Gamma_0}(\omega)\nabla_{\Gamma_0(t)}\hat{v}_\omega) - \sqrt{g_{\Gamma_0}(\omega)}(\alpha \hat{u}_\omega - \beta \hat{v}_\omega) = 0 \quad \text{on } \Gamma_0(t). \tag{3.4.58}$$

We continue by motivating a mean-weak formulation for the ALE coupled-system by the following the derivation presented in [35] for a specific deterministic case. For convenience, we will drop the pull-back notation $\hat{u}$ as all of the subsequent analysis will be over the reference bulk-surface. Let us begin by multiplying the random bulk equation by a smooth test function φ and integrating by parts to obtain

$$\int_{D_0(t)} \partial^\bullet_w(\sqrt{g}u)\varphi + \sqrt{g}u\varphi\nabla \cdot w + \int_{D_0(t)} \sqrt{g}u\nabla\varphi \cdot (w - w_{corr}) + \int_{\Gamma_0(t)} \sqrt{g}u\varphi(w_{corr} - w)\cdot \nu^{\Gamma_0} \tag{3.4.59}$$

$$+ \int_{D_0(t)} \sqrt{g}G^{-1}\nabla u \cdot \nabla \varphi - \int_{\Gamma_0(t)} \varphi\sqrt{g}G^{-1}\nabla u \cdot \nu^{\Gamma_0} = 0. \tag{3.4.60}$$

We observe by Remark 3.4.4 that the third term vanishes since the vector $(w_{corr} - w)$ is tangential to the surface $\Gamma_0(t)$. We now substitute in the boundary condition of the reformulated system to obtain

$$\int_{D_0(t)} \partial^\bullet_w(\sqrt{g}u)\varphi + \sqrt{g}u\varphi\nabla \cdot w + \int_{D_0(t)} \sqrt{g}u\nabla\varphi \cdot (w - w_{corr}) \tag{3.4.61}$$

$$+ \int_{D_0(t)} \sqrt{g}G^{-1}\nabla u \cdot \nabla \varphi + \int_{\Gamma_0(t)} \varphi\sqrt{g_{\Gamma_0}}\varphi(\alpha u - \beta v) = 0. \tag{3.4.62}$$

We next consider the random advection-diffusion equation on the evolving surface $\Gamma_0(t)$. Integrating by parts, recalling that $\partial\Gamma_0(t) = \emptyset$, leads to

$$\int_{\Gamma_0(t)} \partial^\bullet_w(\sqrt{g_{\Gamma_0}}v)\xi + \sqrt{g_{\Gamma_0}}v\xi\nabla_{\Gamma_0(t)} \cdot w + \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}v\nabla_{\Gamma_0(t)}\xi \cdot (w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr}) \\ + \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0}\nabla_{\Gamma_0(t)}v \cdot \nabla_{\Gamma_0(t)}\xi - \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}(\alpha u - \beta v)\xi = 0.$$

Taking a weighted average of the surface and bulk variational formulations gives

$$
\begin{aligned}
&\alpha \int_{D_0(t)} \partial^\bullet_w(\sqrt{g}u)\varphi + \sqrt{g}u\varphi\nabla\cdot w + \alpha\int_{D_0(t)} \sqrt{g}u\nabla\varphi\cdot(w - w_{corr}) + \alpha\int_{D_0(t)} \sqrt{g}G^{-1}\nabla u\cdot\nabla\varphi \\
&+ \beta\int_{\Gamma_0(t)} \partial^\bullet_w(\sqrt{g_{\Gamma_0}}v)\xi + \sqrt{g_{\Gamma_0}}v\xi\nabla_{\Gamma_0(t)}\cdot w + \beta\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}v\nabla_{\Gamma_0(t)}\xi\cdot(w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr}) \\
&+ \beta\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0}\nabla_{\Gamma_0(t)}v\cdot\nabla_{\Gamma_0(t)}\xi + \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}(\alpha u - \beta v)(\alpha\varphi - \beta\xi) = 0.
\end{aligned}
$$

We are now in a position to state the mean-weak formulation for the reformulated coupled system. We first introduce the appropriate random function spaces for the problem. We set for each $t \in [0, t]$

$$
\begin{aligned}
V(t) &= L^2(\Omega; H^1(D_0(t)) \times H^1(\Gamma_0(t))) \\
H(t) &= L^2(\Omega; L^2(D_0(t)) \times L^2(\Gamma_0(t))) \\
V^*(t) &= L^2(\Omega; H^{-1}(D_0(t)) \times H^{-1}(\Gamma_0(t))),
\end{aligned}
$$

and define the push-forward operator on the deterministic function spaces

$$
\phi_t : L^2(D_0(0)) \times L^2(\Gamma_0(0)) \to L^2(D_0(t)) \times L^2(\Gamma_0(t))
$$

component-wise by the associated flow map of the ALE velocity w. Specifically, we define for $u_0 \in L^2(D_0(0))$ and $v_0 \in L^2(\Gamma_0(0))$

$$
\phi_t(u_0, v_0)((\tilde{G}(x,t),t),(\tilde{G}(y,t),t)) = (u_0(x,t), v_0(y,t)) \quad \forall x \in D_0(0) \text{ and } y \in \Gamma_0(0).
$$

By the given smoothness of the flow map $\tilde{G}$ associated to the ALE velocity of the reference bulk-surface stated in assumption 3.4.9, we have that

$$
(L^2(D_0) \times L^2(\Gamma_0), \phi_t) \quad (H^1(D_0) \times H^1(\Gamma_0), \phi_t|_{H^1(D_0(0))\times H^1(\Gamma_0(0))})
$$

are both compatible pairs, and hence by Lemma 3.3.1 we deduce (H, ϕ_t) and $(V, \phi_t|_{V(0)})$ are also compatible. Motivated by the prior derivation of a weak-formulation for the reformulated coupled system, we set the time-dependent bilinear in the abstract framework to be given by

$$
\begin{aligned}
m(t;(u,v),(\varphi,\xi)) &= \alpha\int_\Omega\int_{D_0(t)} \sqrt{g}u\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}v\xi \\
a(t;(u,v),(\varphi,\xi)) &= \alpha\int_\Omega\int_{D_0(t)} \sqrt{g}G^{-1}\nabla u\cdot\nabla\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0}\nabla_{\Gamma_0(t)}v\cdot\nabla_{\Gamma_0(t)}\xi \\
&\quad + \int_\Omega\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}(\alpha u - \beta v)(\alpha\varphi - \beta\xi) \\
b(t;(u,v),(\varphi,\xi)) &= \alpha\int_\Omega\int_{D_0(t)} \sqrt{g}u\nabla\varphi(w - w_{corr}) + \beta\int_\Omega\int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}v\nabla_{\Gamma_0(t)}\xi\cdot(w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr})
\end{aligned}
$$

$$
\begin{aligned}
g(t, w;&(u,v),(\varphi,\xi)) \\
&= \alpha \int_\Omega \int_{D_0(t)} (\partial_w^\bullet(\sqrt{g}) + \sqrt{g}\nabla \cdot w)\, u\varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \left(\partial_w^\bullet(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)} \cdot w\right) v\xi.
\end{aligned}
$$

The mean-weak formulation for Problem 3.4.4 will then reads as follows, where for convenience, will we drop the pull-back notation $\hat{u}$ and instead write u as we will henceforth only be concerned with the formulation on the reference domain.

Problem 3.4.9 (Mean-weak formulation)**.** *Given $(u_0, v_0) \in V_0$, find a pair $(u, v) \in W(V, H)$ such that for a.e. $t \in [0, T]$*

$$
m(t; (\partial_w^\bullet u, \partial_w^\bullet v), (\varphi, \xi)) + g(t, w; (u, v), (\varphi, \xi)) + a(t; (u, v), (\varphi, \xi)) + b(t; (u, v), (\varphi, \xi)) = 0 \quad (3.4.63)
$$

for all $(\varphi, \xi) \in V(t)$, and which furthermore satisfies the initial condition

$$
(u(0), v(0)) = (u_0, v_0) \quad \textit{in } H_0.
$$

By a similar argument as was presented in Lemma 3.4.1 for the uniform estimates on the random coefficients for the reformulated surface advection-diffusion equation, we have the following bounds on the bulk random coefficients, as a result of estimates on the stochastic domain mapping (3.4.38), (3.4.39) and the uniform bound (3.4.28) on the random physical velocity field w_{phys}.

Lemma 3.4.2 (Uniform estimates on bulk coefficients)**.** *There exists constant $C_i > 0$ independent of $\omega \in \Omega$ and $t \in [0, T]$ such that*

$$
\begin{aligned}
C_1 \le \sqrt{g(\omega, t; x)} &\le C_2 & \forall x \in D_0(t) \quad & (3.4.64)\\
C_3|\eta|^2 \le G(\omega, t; x)\eta \cdot \eta &\le C_4|\eta|^2 & \forall x \in D_0(t), \forall \eta \in \mathbb{R}^{n+1} \quad & (3.4.65)\\
|w_{corr}(\omega, t; x)| &\le C_5 & \forall x \in D_0(t). \quad & (3.4.66)
\end{aligned}
$$

Furthermore, such that for all $x \in D_0(t)$

$$
\begin{aligned}
|\partial_w^\bullet(\sqrt{g})(\omega, t; x)| &\le C_6 \quad (3.4.67)\\
|\partial_w^\bullet(G^{-1})(\omega, t; x)| &\le C_7 \quad (3.4.68)\\
|\partial_w^\bullet(w_{corr})(\omega, t; x)| &\le C_8. \quad (3.4.69)
\end{aligned}
$$

Similarly, with the uniform bounds (3.4.37), (3.4.40) on stochastic domain mapping restricting to the surface $\Gamma_0(t)$, and uniform bound (3.4.29) on the tangential component of the random physical velocity field on $\Gamma_\omega(t)$, we have the following estimates for the random surface coefficients.

Lemma 3.4.3 (Uniform estimates on surface coefficients)**.** *There exists constant $C_i > 0$ inde-*

pendent of $\omega \in \Omega$ and $t \in [0,T]$ such that

$$C_1 \le \sqrt{g_{\Gamma_0}(\omega,t;x)} \le C_2 \qquad \forall x \in \Gamma_0(t) \tag{3.4.70}$$
$$C_3|\eta|^2 \le G_{\Gamma_0}(\omega,t;x)\eta\cdot\eta \le C_4|\eta|^2 \qquad \forall x \in \Gamma_0(t), \forall \eta \in T\Gamma_0(t) \tag{3.4.71}$$
$$|w^{\Gamma_0}_{\tau,corr}(\omega,t;x)| \le C_5 \qquad \forall x \in \Gamma_0(t). \tag{3.4.72}$$

Furthermore, such that for all $x \in \Gamma_0(t)$

$$|\partial^\bullet_w(\sqrt{g_{\Gamma_0}})(\omega,t;x)| \le C_6 \tag{3.4.73}$$
$$|\partial^\bullet_w(G^{-1}_{\Gamma_0})(\omega,t;x)| \le C_7 \tag{3.4.74}$$
$$|\partial^\bullet_w(w^{\Gamma_0}_{\tau,corr})(\omega,t;x)| \le C_8. \tag{3.4.75}$$

We identify the additional terms arising in the transport properties (T2) and (T3) for our bilinear forms $a(t;\cdot,\cdot)$ and $b(t;\cdot,\cdot)$ for the coupled problem, with the identity

$$\partial^\bullet_w(\nabla f) = \nabla(\partial^\bullet_w f) - \nabla w^\top \nabla f$$

to be given by

$$\begin{aligned}\tilde{a}(t,w;(u,v),(\varphi,\xi)) = \alpha\int_\Omega\int_{D_0(t)} \tilde{\mathcal{A}}(w)\nabla u\cdot\nabla\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} \tilde{\mathcal{A}}^{\Gamma_0}(w)\nabla_{\Gamma_0(t)}v\cdot\nabla_{\Gamma_0(t)}\xi \\ + \int_\Omega\int_{\Gamma_0(t)} (\alpha u - \beta v)(\alpha\varphi - \beta v)\partial^\bullet_w(\sqrt{g_{\Gamma_0}})\end{aligned}$$

and

$$\tilde{b}(t,w;(u,v),(\varphi,\xi)) = \alpha\int_\Omega\int_{D_0(t)} u\tilde{\mathcal{B}}(w)\cdot\nabla\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} v\tilde{\mathcal{B}}^{\Gamma_0}(w)\cdot\nabla_{\Gamma_0(t)}\xi,$$

where the random coefficients are

$$\begin{aligned}\tilde{\mathcal{A}}(w) &= \partial^\bullet_w(\sqrt{g})G^{-1} + \sqrt{g}\partial^\bullet_w(G^{-1}) + (\nabla\cdot w)\sqrt{g}G^{-1} - 2\sqrt{g}\mathcal{D}(w) \\ \tilde{\mathcal{A}}^{\Gamma_0}(w) &= \partial^\bullet_w(\sqrt{g_{\Gamma_0}})G^{-1}_{\Gamma_0} + \sqrt{g_{\Gamma_0}}\partial^\bullet_w(G^{-1}_{\Gamma_0}) + (\nabla_{\Gamma_0(t)}\cdot w)\sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0} - 2\sqrt{g_{\Gamma_0}}\mathcal{D}^{\Gamma_0}(w)\end{aligned}$$

with the deformation tensors

$$\mathcal{D}(w) = \frac{1}{2}\left(G^{-1}\nabla w^\top + \nabla w G^{-1}\right) \tag{3.4.76}$$
$$\mathcal{D}^{\Gamma_0}(w) = \frac{1}{2}\left(G^{-1}_{\Gamma_0}\nabla_{\Gamma_0(t)}w^\top + \nabla_{\Gamma_0(t)}wG^{-1}_{\Gamma_0}\right) \tag{3.4.77}$$

and with

$$\begin{aligned}\tilde{\mathcal{B}}(w) &= (\partial^\bullet_w(\sqrt{g}) - \nabla w + \sqrt{g}\nabla\cdot w)(w - w_{corr}) + \sqrt{g}\partial^\bullet_w(w - w_{corr}) \\ \tilde{\mathcal{B}}^{\Gamma_0}(w) &= \left(\partial^\bullet_w(\sqrt{g_{\Gamma_0}}) - \nabla_{\Gamma_0(t)}w + \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)}\cdot w\right)(w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr}) + \sqrt{g_{\Gamma_0}}\partial^\bullet_w(w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr}).\end{aligned}$$

We may now employ the uniform bounds on the random coefficients and their material derivatives, given in Lemma 3.4.2 and Lemma 3.4.3, to deduce that the bilinear forms $g(t, w; \cdot, \cdot)$, $\tilde{a}(t, w; \cdot, \cdot)$ and $\tilde{b}(t, w; \cdot, \cdot)$ are uniformly bounded. Hence all the assumptions listed in the abstract analysis are satisfied and thus we deduce the existence and uniqueness of the solution $(u, v) \in W(V, H)$ to the mean-weak formulation, Problem 3.4.9 for the reformulated coupled system. Furthermore, the weak solution satisfy the energy estimate

$$\sup_{t\in[0,T]} \|(u(t), v(t))\|^2_{H(t)} + \int_0^T \|(u(t), v(t))\|^2_{V(t)}\, dt \lesssim \|(u_0, v_0)\|^2_{H_0}$$

$$\int_0^T \|(\partial^\bullet_w u(t), \partial^\bullet_w v(t))\|^2_{H(t)}\, dt + \sup_{t\in[0,T]} \|(u(t), v(t))\|^2_{V(t)} \lesssim \|(u_0, v_0)\|^2_{V_0}.$$

3.5 Abstract numerical analysis

In this section, we derive optimal order error bounds for an evolving finite element discretisation based upon a first order approximation of the curved geometry, coupled with a Monte-Carlo sampling approximation. The analysis of the evolving surface finite element method has been well-analysed, with energy estimates first derived in [30], L^2-estimates subsequently derived in [31], further error bounds for an ALE scheme established in [40], and finally a fully unified theory for higher order evolving elements in [35], which will serve as a basis for the subsequent section. We will now continue by outlining the unified framework presented in [35] adapted to our random function spaces and for the specific case of linear finite elements. We will then present all the necessary assumptions listed in [35] to derive optimal error bounds, which will similarly be adapted to our random context, and combine the errors bounds in [35] with estimates on the Monte-Carlo approximation, to obtain error estimates for the fully discrete scheme proposed.

3.5.1 Abstract semi-discrete problem

Let us begin by first introducing an abstract setting for the general form of a semi-discretisation (spatially) of the continuous mean-weak formulation. We will have at each time $t \in [0, T]$, a family of real separable Hilbert spaces

$$V_h(t) \subset H_h(t)$$

with $h \in (0, h_0)$ denoting the spatial discretisation parameter, which respectively represent discrete approximations of the smooth random function spaces

$$V(t) = L^2(\Omega, \mathcal{V}(t)) \qquad H(t) = L^2(\Omega, \mathcal{H}(t))$$

with realisations taking values in a chosen evolving finite element space. More precisely, the discrete random spaces are given by

$$V_h(t) = L^2(\Omega; \mathcal{V}_h(t)) \qquad H_h(t) = L^2(\Omega; \mathcal{V}_h(t))$$

with $\mathcal{V}_h(t)$ denoting the chosen finite element space defined on the evolving discrete domain which approximates the smooth reference domain. The finite element space $\mathcal{V}_h(t)$ in the abstract setting, is equipped with two norms $\|\cdot\|_{\mathcal{V}_h(t)}$ and $\|\cdot\|_{\mathcal{H}_h(t)}$ which are assumed to satisfy

$$\|\Phi_h\|_{\mathcal{H}_h(t)} \leq \|\Phi_h\|_{\mathcal{V}_h(t)} \qquad \forall \Phi_h \in \mathcal{V}_h(t)$$

and we denote the corresponding normed spaces by

$$\mathcal{V}_h(t) = (\mathcal{V}_h(t), \|\cdot\|_{\mathcal{V}_h(t)}) \qquad \mathcal{H}_h(t) = (\mathcal{V}_h(t), \|\cdot\|_{\mathcal{H}_h(t)}).$$

The subsequent norms on the random function spaces $V_h(t)$ and $H_h(t)$ are then defined by

$$\|\Phi_h\|^2_{V_h(t)} = \int_\Omega \|\Phi_h(\omega)\|^2_{\mathcal{V}_h(t)} \qquad \|\Phi_h\|^2_{H_h(t)} = \int_\Omega \|\Phi_h(\omega)\|^2_{\mathcal{H}_h(t)}.$$

We assume that the evolution of the discrete finite element space $\mathcal{V}_h(t)$ over time, is described by a given push-forward operator

$$\phi^h_t : \mathcal{V}_h(0) \to \mathcal{V}_h(t)$$

which is such that both of the pairs $(\mathcal{H}_h, \phi^h_t)$ and $(\mathcal{V}_h, \phi^h_t)$ are uniformly compatible in h. More precisely, we assume the constants $C_i > 0$ appearing in the estimates

$$C_1\|\Phi_h\|_{\mathcal{H}_h(0)} \leq \|\phi^h_t \Phi_h\|_{\mathcal{H}_h(t)} \leq C_2\|\Phi_h\|_{\mathcal{H}_h(0)} \quad \forall \Phi_h \in \mathcal{V}_h(0) \tag{3.5.1}$$

$$C_3\|\Phi_h\|_{\mathcal{V}_h(0)} \leq \|\phi^h_t \Phi_h\|_{\mathcal{V}_h(t)} \leq C_4\|\Phi_h\|_{\mathcal{V}_h(0)} \quad \forall \Phi_h \in \mathcal{V}_h(0), \tag{3.5.2}$$

are independent of the spatial discretisation parameter h. We denote the strong material derivative of functions $\Phi_h \in C^1_{\mathcal{H}_h}$, with respect to the push-forward operator ϕ^h_t by $\partial^\bullet_h \Phi_h$.

Remark 3.5.1 (Discrete push-forward operator). *In applications, the discrete push-forward operator ϕ^h_t for the abstract finite element space $\mathcal{V}_h(t)$, will be given by the flow map associated with the discrete velocity V_h of the discrete domain. That is, the abstract strong material derivative will be given by $\partial^\bullet_h \Phi_h = \partial^\bullet_{V_h} \Phi_h$.*

The push-forward operator may now be extended to the random discrete function spaces $V_h(t)$ and $H_h(t)$ pathwise as previously discussed in section 3.3.2, and we have h-uniform compatibility of the pairs $(H_h(t), \phi^h_t)$ and $(V_h(t), \phi^h_t|_{V(0)})$ as a result of Lemma 3.3.1. The discrete analogues of the continuous time-dependent bilinear forms given in the mean-weak formulation, defined over the evolving discrete domain will be denote by

$$
\begin{aligned}
m_h(t;\cdot,\cdot) &: H_h(t)\times H_h(t)\to\mathbb{R}\\
a_h(t;\cdot,\cdot) &: V_h(t)\times V_h(t)\to\mathbb{R}\\
b_h(t;\cdot,\cdot) &: H_h(t)\times V_h(t)\to\mathbb{R},
\end{aligned}
$$

and will be assumed to satisfy the following assumptions. Note that all the subsequent constants $C>0$ appearing the estimates below will be assumed to be independent of $h\in(0,h_0)$.

Assumption 3.5.1 (Discrete uniform bounds)**.** *We assume that $m_h(t;\cdot,\cdot)$ and $a_h(t;\cdot,\cdot)$ are both symmetric*

$$
m_h(t;\Phi_h,\Psi_h)=m_h(t;\Psi_h,\Phi_h)\qquad a_h(t;\Phi_h,\Psi_h)=a_h(t;\Psi_h,\Phi_h)
$$

and that there exists constants $C_i>0$ independent of $t\in[0,T]$ and $h\in(0,h_0)$ such that

$$
\begin{aligned}
C_1\|\Phi_h\|^2_{H_h(t)} \le m_h(t;\Phi_h,\Phi_h) &\le C_2\|\Phi_h\|^2_{H_h(t)} && \forall\Phi_h\in H_h(t) && (M_h1)\\
|a_h(t;\Phi_h,\Psi_h)| &\le C_3\|\Phi_h\|_{V_h(t)}\|\Psi_h\|_{V_h(t)} && \forall\Phi_h,\Psi_h\in V_h(t), && (A_h1)\\
a_h(t;\Phi_h,\Phi_h) &\ge C_4\|\Phi_h\|^2_{V_h(t)}-C_5\|\Phi_h\|^2_{H_h(t)} && \forall\Phi_h\in V_h(t), && (A_h2)\\
|b_h(t;\Phi_h,\Psi_h)| &\le C_6\|\Phi_h\|_{H_h(t)}\|\Psi_h\|_{V_h(t)} && \forall\Phi_h\in H_h(t),\Psi_h\in V_h(t). && (B_h1)
\end{aligned}
$$

Assumption 3.5.2 (Discrete transport property)**.** *We further assume that there exists time-dependent bilinear forms*

$$
\begin{aligned}
g_h(t,V_h;\cdot,\cdot) &: H_h(t)\times H_h(t)\to\mathbb{R}\\
\tilde a_h(t,V_h;\cdot,\cdot) &: V_h(t)\times V_h(t)\to\mathbb{R}\\
\tilde b_h(t,V_h;\cdot,\cdot) &: H_h(t)\times V_h(t)\to\mathbb{R},
\end{aligned}
$$

such that the following abstract transport properties

$$
\begin{aligned}
\frac{d}{dt}m_h(t;\Phi_h,\Psi_h) &= m_h(t;\partial_h^\bullet\Phi_h,\Psi_h)+m_h(t;\Phi_h,\partial_h^\bullet\Psi_h)+g_h(t,V_h;\Phi_h,\Psi_h) && (T_h1)\\
\frac{d}{dt}a_h(t;\Phi_h,\Psi_h) &= a_h(t;\partial_h^\bullet\Phi_h,\Psi_h)+a_h(t;\Phi_h,\partial_h^\bullet\Psi_h)+\tilde a_h(t,V_h;\Phi_h,\Psi_h) && (T_h2)\\
\frac{d}{dt}b_h(t;\Phi_h,\Psi_h) &= b_h(t;\partial_h^\bullet\Phi_h,\Psi_h)+b_h(t;\Phi_h,\partial_h^\bullet\Psi_h)+\tilde b_h(t,V_h;\Phi_h,\Psi_h), && (T_h3)
\end{aligned}
$$

hold for all $\Phi_h,\Psi_h\in C^1_{H_h},C^1_{V_h}$, where the time derivative exists classically.

Assumption 3.5.3 (Uniform bounds on time-derivative)**.** *We assume that there exists con-*

stants $C_i > 0$ independent of $t \in [0,T]$, such that

$$|g_h(t, V_h; \Phi_h, \Psi_h)| \leq C_1 \|\Phi_h\|_{H(t)} \|\Psi_h\|_{H_h(t)} \qquad \forall \Phi_h, \Psi_h \in H_h(t) \tag{M_h2}$$

$$|\tilde{a}_h(t, V_h; \Phi_h, \Psi_h)| \leq C_2 \|\Phi_h\|_{V_h(t)} \|\Psi_h\|_{V_h(t)} \qquad \forall \Phi_h, \Psi_h \in V_h(t) \tag{A_h3}$$

$$|\tilde{b}_h(t, V_h; \Phi_h, \Psi_h)| \leq C_3 \|\Phi_h\|_{H_h(t)} \|\Psi_h\|_{V_h(t)} \qquad \forall \Phi_h \in H_h(t), \Psi_h \in V_h(t). \tag{B_h2}$$

We are now almost in a position to state the general form of a semi-discretisation of the mean-weak formulation in the abstract setting. It just remains to define a suitable solution space for the semi-discrete problem. For this, we introduce the space

$$\tilde{C}^1_{V_h} = \{U_h(\omega, t) = \sum_{j=1}^{N} \alpha_j(\omega, t) \chi_j^t \,|\, \alpha_j \in L^2(\Omega \times [0,T]),\ \alpha_j(\omega, \cdot) \in A.C.([0,T]) \text{ for a.e. } \omega\}$$

where here we let $\{\chi_j^0\}_{j=1}^N \subset \mathcal{V}_h(0)$ denote a basis of the initial finite element space $\mathcal{V}_h(0)$ and set

$$\chi_j^t = \phi_h^t \chi_j^0$$

to be its corresponding push-forward onto $\mathcal{V}_h(t)$. Since the mapping $\phi_h^t : \mathcal{V}_h(0) \to \mathcal{V}_h(t)$ is bijective, it follows that $\{\chi_j^t\}_{j=1}^N$ forms a basis of $\mathcal{V}_h(t)$. Furthermore, by construction of χ_j^t we have $\partial_h^\bullet \chi_j^t = 0$ for all $j = 1, ..., N$. Consequently, the strong material derivative of a function belonging to the solution space $U_h \in \tilde{C}^1_{V_h}$ may be expresssed as

$$\partial_h^\bullet U_h(\omega, t) = \sum_{j=1}^{N} \partial_t \alpha(\omega, t) \chi_j^t.$$

The semi-discrete problem in the abstract framework will now reads as follows.

Problem 3.5.1 (Abstract semi-discrete problem)**.** *Given $U_{h,0} \in H_h(0)$, find $U_h \in \tilde{C}^1_{V_h}$ such that for a.e. $t \in [0,T]$,*

$$m_h(t; \partial_h^\bullet U_h, \Phi_h) + g_h(t, V_h; U_h, \Phi_h) + a_h(t; U_h, \Phi_h) + b_h(t; U_h, \Phi_h) = 0 \tag{3.5.3}$$

for all $\Phi_h \in V_h(t)$ and which furthermore satisfies the initial condition

$$U_h(0) = U_{h,0} \quad \text{in } H_h(0).$$

Since the discrete bilinear forms satisfy equivalent assumptions to their continuous counterparts, we may derive a unique weak solution to the semi-discrete problem in the Hilbert space $U_h \in W(V_h, H_h)$ following a Galerkin-type argument. In particular, as $U_h \in L^2_{V_h}$, and thus

$$\phi_h^{-(\cdot)} U_h(\cdot) \in L^2(0,T; L^2(\Omega; \mathcal{V}_h(0))) \cong L^2(\Omega \times [0,T]) \otimes \mathcal{V}_h(0)$$

by the tensor structure provided in Theorem A.3.2, we deduce that the weak solution is of the form

$$U_h(\omega, t) = \sum_{j=1}^{N} \alpha_j(\omega, t)\chi_j^t \tag{3.5.4}$$

with $\alpha_j(\cdot,\cdot) \in L^2(\Omega \times [0,T])$ for $j = 1, ..., N$. It thus remains to show that realisations of the random coefficients $\{\alpha_j\}$ of the discrete solution are absolutely continuous on $[0,T]$. For this, we first note that the abstract discrete bilinear forms in the semi-discrete formulation are of a particular form as is the case for the continuous setting. More precisely, the bilinear forms may be expressed as

$$\begin{aligned}
m_h(t; U_h, \Phi_h) &= \int_\Omega m_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) \\
a_h(t; U_h, \Phi_h) &= \int_\Omega a_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) \\
b_h(t; U_h, \Phi_h) &= \int_\Omega b_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) \\
g_h(t, V_h; U_h, \Phi_h) &= \int_\Omega g_h(t, \omega, V_h; U_h(\omega, t), \Phi_h(\omega, t))
\end{aligned}$$

for given sample-dependent bilinears forms

$$\begin{aligned}
m_h(t, \omega; \cdot, \cdot) &: \mathcal{H}_h(t) \times \mathcal{H}_h(t) \to \mathbb{R} \\
a_h(t, \omega; \cdot, \cdot) &: \mathcal{V}_h(t) \times \mathcal{V}_h(t) \to \mathbb{R} \\
b_h(t, \omega; \cdot, \cdot) &: \mathcal{H}_h(t) \times \mathcal{V}_h(t) \to \mathbb{R} \\
g(t, \omega, V_h; \cdot, \cdot) &: \mathcal{H}_h(t) \times \mathcal{H}_h(t) \to \mathbb{R}.
\end{aligned}$$

We assume that the above sample-dependent bilinear forms satisfy equivalent conditions as are imposed on the time-dependent bilinear forms above, where in particular, we assume that all the constants appearing in the equivalent estimates are independent of ω as well as $t \in [0,T]$. Note that although it is sufficient to impose the uniform assumptions on the sample-dependent bilinear forms and to then derive the equivalent properties for the time-dependent bilinear forms, we avoid this approach for the sake of brevity and relisting assumptions, particularly as we only require these particular properties of the sample-dependent bilinear forms for the subsequent proof. We may now re-express the semi-discrete formulation as follows

$$\begin{aligned}
\int_\Omega m_h(t, \omega; \partial_h^\bullet U_h(\omega, t), \Phi_h(\omega, t)) + \int_\Omega g_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) \\
+ \int_\Omega a_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) + \int_\Omega b_h(t, \omega; U_h(\omega, t), \Phi_h(\omega, t)) = 0.
\end{aligned}$$

Substituting the discrete solution (3.5.4) into the semi-discrete problem and choosing the test function $\Phi_h \in V(t) = L^2(\Omega; \mathcal{V}_h(t))$ to be given by $\Phi_h = \chi_k^t \phi_k(\omega)$ for an arbitrary $\phi_k \in L^2(\Omega)$, leads to the equivalent problem of finding $\alpha(\omega,t) = (\alpha_1(\omega,t),...,\alpha_N(\omega,t)) \in \mathbb{R}^N$ such that

$$\int_\Omega M(\omega,t)\alpha_t(\omega,t)\cdot\Phi(\omega) + (G(\omega,t)+S(\omega,t)+B(\omega,t))\,\alpha(\omega,t)\cdot\Phi(\omega) = 0$$

for all $\Phi(\omega) = (\Phi_1(\omega),...,\Phi_N(\omega)) \in L^2(\Omega;\mathbb{R}^N)$, where we have for $j,k = 1,...,N$,

$$M_{j,k}(\omega,t) = m(t,\omega;\chi_j^t,\chi_k^t) \tag{3.5.5}$$

$$G_{j,k}(\omega,t) = g(t,\omega,V_h;\chi_j^t,\chi_k^t) \tag{3.5.6}$$

$$A_{j,k}(\omega,t) = a(t,\omega;\chi_j^t,\chi_k^t) \tag{3.5.7}$$

$$B_{j,k}(\omega,t) = b(t,\omega;\chi_j^t,\chi_k^t). \tag{3.5.8}$$

Since $\Phi(\omega)$ is arbitrary, we deduce that the semi-discrete problem is equivalent to finding for $a.e.\omega \in \Omega$, $\alpha(\omega,\cdot) : [0,T] \to \mathbb{R}^N$ which solves the system of ODEs

$$M(\omega,t)\alpha_t(\omega,t) + (G(\omega,t)+S(\omega,t)+B(\omega,t))\,\alpha(\omega,t) = 0,$$

subject to an initial condition. As $M(\omega,\cdot)$ is positive-definite and inverse is uniformly bounded $M(\omega,\cdot)^{-1} \in L^\infty(0,T;\mathbb{R}^{N\times N})$ by assumptions (M_h1), and furthermore that $G(\omega,\cdot), S(\omega,\cdot)$ and $B(\omega,\cdot) \in L^\infty(0,T;\mathbb{R}^{N\times N})$ by assumptions (M_h2), (A_h1) and (B_h1), we deduce from standard ODE theory [19], the existence of a unique $\alpha(\omega,\cdot) \in A.C.[0,T]$ which solves the system of ODEs for $a.e.$ $t \in [0,T]$. We have therefore established the following well-posedness result of the semi-discrete problem and we note that the stability estimates may be derived in a similar manner to the continuous problem.

Theorem 3.5.1 (Well-posedness)**.** *Given $U_{h,0} \in V_h(0)$, there exists a unique solution $U_h \in \tilde{C}_{V_h}$ to the semi-discrete problem which furthermore satisfies the following stability estimates*

$$\sup_{t\in[0,T]} \|U_h(t)\|^2_{H_h(t)} + \int_0^T \|U_h(t)\|^2_{V_h(t)}\,dt \lesssim \|U_{h,0}\|^2_{H_h(0)} \tag{3.5.9}$$

$$\int_0^T \|\partial_h^\bullet U_h(t)\|^2_{H_h(t)}\,dt + \sup_{t\in[0,T]} \|U_h(t)\|^2_{V_h(t)} \lesssim \|U_{h,0}\|^2_{V_h(0)}. \tag{3.5.10}$$

3.5.2 Abstract error analysis

We now proceed by stating all the necessary assumptions required in order to derive optimal order error estimates for the semi-discretisation of the continuous mean-weak formulation. Let us first begin by introducing the Hilbert space $\mathcal{Z}(t)$, which will represent in the abstract setting the subspace of higher regularity functions contained within $\mathcal{Z}(t) \subset \mathcal{V}(t)$, for which an interpolation estimate is available. We additionally introduce the spaces $\mathcal{Z}_0(t)$ and $\mathcal{Z}_{0,h}(t)$, which will respectively represent the space of continuous functions defined over the smooth reference domain and the discrete approximating domain, and will assume $\mathcal{V}_h(t) \subset \mathcal{Z}_{0,h}(t)$ and $\mathcal{Z}(t) \subset \mathcal{Z}_0(t)$. For these abstract spaces, we assume there will exists a lifting mapping as follows.

Assumption 3.5.4 (Lifting mapping)**.** *We assume that there exists a bijective linear mapping at each time $t \in [0,T]$*

$$\Lambda_h(\cdot,t) : \mathcal{Z}_{0,h}(t) \to \mathcal{Z}_0(t), \tag{3.5.11}$$

and furthermore that there exists constants $C_i > 0$ independent of $h > 0$ and $t \in [0,T]$ such that for all $\Phi_h \in \mathcal{Z}_{0,h}(t)$ with corresponding lift $\varphi_h = \Lambda(\Phi_h, t) \in \mathcal{Z}_0(t)$, we have the following estimates whenever the norms exists

$$C_1\|\Phi_h\|_{\mathcal{H}_h(t)} \leq \|\varphi_h\|_{\mathcal{H}(t)} \leq C_2\|\Phi_h\|_{\mathcal{H}_h(t)} \tag{L1}$$

$$C_3\|\Phi_h\|_{\mathcal{V}_h(t)} \leq \|\varphi_h\|_{\mathcal{V}(t)} \leq C_4\|\Phi_h\|_{\mathcal{V}_h(t)}. \tag{L2}$$

We denote the lift of a function $\Phi_h \in \mathcal{Z}_{0,h}(t)$ by Φ_h^l, and the inverse lift of a function $\varphi \in \mathcal{Z}_0(t)$ by φ^{-l}. In order to compare the lifted discrete solution with the continuous solution of the mean-weak formulation, we impose the following assumption on the lifted finite element space.

Assumption 3.5.5 (Lifted finite element space)**.** *We assume the image of the finite element space $\mathcal{V}_h(t)$ under the lifting mapping, i.e. $\mathcal{V}_h^l(t) = \Lambda_h(\mathcal{V}_h(t), t)$ is contained within $\mathcal{V}_h^l(t) \subset \mathcal{V}(t)$.*

Naturally, the lifted finite element space has by construction an induced flow map describing its evolution from the initial finite element space $\phi_t^l : \mathcal{V}_h^l(0) \to \mathcal{V}_h^l(t)$ defined by

$$\phi_t^l(u_0^l) = (\phi_t u_0)^l \qquad \forall u_0 \in \mathcal{V}_h(0).$$

We assume that the induced discrete material flow map ϕ_t^l, defines the evolution of a much wider class of functions, as suppose to just the lifted finite element space.

Assumption 3.5.6 (Discrete material flow)**.** *We assume that there exists an extension of the push-forward operator $\phi_t^l : \mathcal{V}_h^l(0) \to \mathcal{V}_h^l(t)$ to the space*

$$\phi_t^l : \mathcal{H}(0) \to \mathcal{H}(t) \tag{3.5.12}$$

such that both of the pairs $(\mathcal{H}, \phi_t^l)$ and $(\mathcal{V}, \phi_t^l|_{\mathcal{V}(0)})$ are uniformly compatible in h, see A.1.1 for further details on compatability.

We shall denote the strong material derivative associated with the flow map ϕ_t^l of functions $f \in C^1_{\mathcal{H}}$ by

$$\partial_h^\bullet f(t) = \phi_t^l\left(\frac{d}{dt}(\phi_{-t}^l f(t))\right).$$

Remark 3.5.2 (Abstract lifted flow map ϕ_t^l)**.** *In applications, the extended lifted flow map ϕ_t^l will relate the material flow on the smooth domain, induced by the discrete material velocity on the discrete domain, under the given lifting mapping. In particular, the abstract strong material derivative $\partial_h^\bullet f$ with respect to the push-forward operator (3.5.12), will given by $\partial_{v_h}^\bullet f$, with v_h denoting the induced velocity field on the smooth domain.*

We now continue by defining extensions of the lifting mapping $\Lambda_h(\cdot, t)$ and push-forward operator ϕ_t^l to their corresponding random function spaces. For the lifting map, we define an extension

$$\Lambda(\cdot, t) : Z_{0,h}(t) \to Z_0(t)$$

to the random spaces $Z_{0,h}(t) = L^2(\Omega; \mathcal{Z}_{0,h}(t))$ and $Z_0(t) = L^2(\Omega; \mathcal{Z}_0(t))$ pathwise by

$$(u_{0,h})^l(\omega) = (u_{0,h}(\omega))^l \qquad \forall u_{0,h} \in Z_{0,h}(t)$$

and will denote the lift of the random discrete space $V_h(t) = L^2(\Omega, \mathcal{V}_h^l(t))$ by $V_h^l(t)$. Here, we note that $V_h^l(t) \subset V(t)$. We similarly define an extension of the lifted push-forward operator $\phi_t^l : \mathcal{H}(0) \to \mathcal{H}(t)$ to a mapping

$$\phi_t^l : H(0) \to H(t)$$

with $H(t) = L^2(\Omega; \mathcal{H}(t))$ pathwise. By our assumptions (L1) and (L2) on the deterministic lifting mapping and the compatibility assumption on the push-forward operator ϕ_t^l given in Assumption 3.5.6, we deduce the following equivalent results for their random extensions.

Lemma 3.5.1. *The lifting mapping $\Lambda_h(\cdot, t) : Z_{0,h}(t) \to Z_0(t)$ satisfies the following estimates*

$$C_1\|\Phi_h\|_{H_h(t)} \le \|\varphi_h\|_{H(t)} \le C_2\|\Phi_h\|_{H_h(t)}$$
$$C_3\|\Phi_h\|_{V_h(t)} \le \|\varphi_h\|_{V(t)} \le C_4\|\Phi_h\|_{V_h(t)},$$

for constants $C_i > 0$ independent of $t \in [0, T]$ whenever the above norms exists. Furthermore, the pushforward operator $\phi_t^l : H(0) \to H(t)$ is such that both of the pairs (H, ϕ_t^l) and $(V, \phi_t^l|_{V(0)})$ are uniformly compatible in h.

We may now introduce the discrete transport properties with respect to the discrete material velocity v_h for the continuous abstract bilinear forms.

Assumption 3.5.7 (Discrete lifted transport property)**.** *We assume that there exists time-dependent bilinear forms*

$$g(t, v_h; \cdot, \cdot) : H(t) \times H(t) \to \mathbb{R}$$
$$\tilde{a}(t, v_h; \cdot, \cdot) : V(t) \times V(t) \to \mathbb{R}$$
$$\tilde{b}(t, v_h; \cdot, \cdot) : H(t) \times V(t) \to \mathbb{R},$$

such that we have the following abstract transport properties

$$\frac{d}{dt}m(t; \varphi, \psi) = m(t; \partial_h^\bullet \varphi, \psi) + m(t; \varphi, \partial_h^\bullet \psi) + g(t, v_h; \varphi, \psi) \quad \forall \varphi, \psi \in C_H^1 \qquad (T_h^l 1)$$

$$\frac{d}{dt}a(t; \varphi, \psi) = a(t; \partial_h^\bullet \varphi, \psi) + a(t; \varphi, \partial_h^\bullet \psi) + \tilde{a}(t, v_h; \varphi, \psi) \quad \forall \varphi, \psi \in C_V^1 \qquad (T_h^l 2)$$

$$\frac{d}{dt}b(t; \varphi, \psi) = b(t; \partial_h^\bullet \varphi, \psi) + b(t; \varphi, \partial_h^\bullet \psi) + \tilde{b}(t, v_h; \varphi, \psi) \quad \forall \varphi \in C_H^1, \psi \in C_V^1, \qquad (T_h^l 3)$$

where the above derivative exists classically.

We assume that the above bilinear forms are uniformly bounded as follows.

Assumption 3.5.8 (Uniform bounds on derivatives)**.** *We assume there exists constants $C_i > 0$ independent of $t \in [0,T]$ and $h > 0$ such that we have*

$$|g(t, v_h; \varphi, \psi)| \leq C_1 \|\varphi\|_{H(t)} \|\psi\|_{H(t)} \qquad \forall \varphi, \psi \in H(t) \tag{M^l2}$$

$$|\tilde{a}(t, v_h; \varphi, \psi)| \leq C_2 \|\varphi\|_{V(t)} \|\psi\|_{V(t)} \qquad \forall \varphi, \psi \in V(t) \tag{A^l3}$$

$$|\tilde{b}(t, v_h; \varphi, \psi)| \leq C_3 \|\varphi\|_{H(t)} \|\psi\|_{V(t)} \qquad \forall \varphi \in H(t), \psi \in V(t). \tag{B^l2}$$

We next state an interpolation estimate for the deterministic Hilbert space $\mathcal{Z}(t) \subset \mathcal{V}(t)$ consisting of functions of higher regularity.

Assumption 3.5.9 (Interpolation estimate)**.** *We assume that there exists a well-defined operator $I_h : \mathcal{Z}(t) \to \mathcal{V}_h^l(t)$ at each time $t \in [0,T]$, such that we have the following estimate for a constant $C > 0$ independent of h and $t \in [0,T]$*

$$\|\eta - I_h \eta\|_{\mathcal{H}(t)} + h \|\eta - I_h \eta\|_{\mathcal{V}(t)} \leq C h^2 \|\eta\|_{\mathcal{Z}(t)} \quad \forall \eta \in \mathcal{Z}(t). \tag{I}$$

We may extend the interpolation operator $I_h : Z(t) \to V_h^l(t)$ pathwise

$$(I_h u_0)(\omega) = I_h(u_0(\omega)) \qquad \forall u_0 \in L^2(\Omega, \mathcal{Z}(t)) =: Z(t)$$

and consequently have the following estimates by (I).

Lemma 3.5.2 (Interpolation estimate for random spaces)**.** *There exists a constant $C > 0$ independent of $t \in [0,T]$ such that for all $\eta \in Z(t)$ we have*

$$\|\eta - I_h \eta\|_{H(t)} + h \|\eta - I_h \eta\|_{V(t)} \leq C h^2 \|\eta\|_{Z(t)} \quad \forall \eta \in Z(t).$$

We now impose bounds on the errors arising from the geometric pertubation of the continuous bilinear forms. Note that these estimates will be based on a piece-wise linear approximation of the curved evolving reference domain.

Assumption 3.5.10 (Geometric pertubations)**.** *We assume that there exists constants $C > 0$ independent of h and $t \in [0,T]$ such that for all $\Phi_h, \Psi_h \in V_h(t), H_h(t)$ with respective lifts φ_h, ψ_h, we have*

$$|m(t; \varphi_h, \psi_h) - m_h(t; \Phi_h, \Psi_h)| \leq C h^2 \|\varphi_h\|_{H(t)} \|\psi_h\|_{H(t)} \tag{G1}$$

$$|a(t; \varphi_h, \psi_h) - a_h(t; \Phi_h, \Psi_h)| \leq C h \|\varphi_h\|_{V(t)} \|\psi_h\|_{V(t)} \tag{G2}$$

$$|b(t; \varphi_h, \psi_h) - b_h(t; \Phi_h, \Psi_h)| \leq C h \|\varphi_h\|_{H(t)} \|\psi_h\|_{V(t)} \tag{G3}$$

and

$$|g(t, v_h; \varphi_h, \psi_h) - g_h(t, V_h; \Phi_h, \Psi_h)| \leq Ch^2 \|\varphi_h\|_{H(t)} \|\psi_h\|_{H(t)} \tag{G4}$$
$$|\tilde{a}(t, v_h; \varphi_h, \psi_h) - \tilde{a}_h(t, V_h; \Phi_h, \Psi_h)| \leq Ch \|\varphi_h\|_{V(t)} \|\psi_h\|_{V(t)} \tag{G5}$$
$$|\tilde{b}(t, v_h; \varphi_h, \psi_h) - \tilde{b}_h(t, V_H; \Phi_h, \Psi_h)| \leq Ch \|\varphi_h\|_{H(t)} \|\psi_h\|_{V(t)}. \tag{G6}$$

Remark 3.5.3 (Geometric error). *In the case of curved surfaces, the above estimates will relate to the error due to the geometric approximation of the continuous bilinear forms defined over the smooth surface onto the (piece-wise linear) discrete surface, and thus will be of the order h^2. However, for the case of curved bulk domains, the above perturbation estimates will comprise of a geometric error of order h over boundary simplices which are contained within neighbourhood of the curved boundary of width order h. To attain higher order estimates for the geometric pertubation will necessitate the use of a narrow band trace inequality which may require higher regularity on the given test functions Φ_h and Ψ_h.*

We characterise the higher order estimates for the above geometric pertubations of the bilinear forms restricted to the abstract random function space $Z(t)$, consisting of functions whose realisations are of a higher regularity, as follows.

Assumption 3.5.11 (Higher order geometric pertubations). *Given $\eta, \varphi \in Z(t)$, with inverse lift η^{-l}, φ^{-l} we have*

$$|a(t; \varphi, \psi) - a_h(t; \varphi^{-l}, \psi^{-l})| \leq Ch^2 \|\varphi\|_{Z(t)} \|\psi\|_{Z(t)} \tag{G7}$$
$$|b(t; \varphi, \psi) - b_h(t; \varphi^{-l}, \psi^{-l})| \leq Ch^2 \|\varphi\|_{Z(t)} \|\psi\|_{Z(t)} \tag{G8}$$
$$|\tilde{a}(t, v_h; \varphi, \psi) - \tilde{a}_h(t, V_h; \varphi^{-l}, \psi^{-l})| \leq Ch^2 \|\varphi\|_{Z(t)} \|\psi\|_{Z(t)} \tag*{(G8)}$$
$$|\tilde{b}(t, v_h; \varphi, \psi) - \tilde{b}_h(t, V_h; \varphi^{-l}, \psi^{-l})| \leq Ch^2 \|\varphi\|_{Z(t)} \|\psi\|_{Z(t)}. \tag{G9}$$

We assume the errors arising in our variational set up due to the discrete approximation of the smooth velocity are bounded as follows.

Assumption 3.5.12 (Discrete velocity estimate). *There exists constants $C > 0$ independent of h and $t \in [0, T]$ such that we have the following estimates*

$$\|\partial_h^\bullet \varphi - \partial^\bullet \varphi\|_{H(t)} \leq Ch^2 \|\varphi\|_{V(t)} \qquad \forall \varphi \in V(t) \tag{V1}$$
$$\|\partial_h^\bullet \varphi - \partial^\bullet \varphi\|_{V(t)} \leq Ch \|\varphi\|_{Z(t)} \qquad \forall \varphi \in Z(t). \tag{V2}$$

Furthermore, we have

$$|\tilde{a}(t, v; \varphi_h, \psi_h) - \tilde{a}(t, v_h; \varphi_h, \psi_h)| \leq ch \|\varphi_h\|_{V(t)} \|\psi_h\|_{V(t)} \quad \forall \varphi_h, \psi_h \in V_h^l(t) \tag{V3}$$
$$|\tilde{b}(t, v; \varphi_h, \psi_h) - \tilde{b}(t, v_h; \varphi_h, \psi_h)| \leq ch \|\varphi_h\|_{H(t)} \|\psi_h\|_{V(t)} \quad \forall \varphi_h \in H_h^l(t) \text{ and } \psi_h \in V_h^l(t). \tag{V4}$$

We finally conclude by introducing an associated dual problem. We take $\kappa > 0$ sufficiently large such that the modified bilinear form $a_\kappa(t;\cdot,\cdot) : V(t) \times V(t) \to \mathbb{R}$ defined by

$$a_\kappa(t;\cdot,\cdot) = a(t;\cdot,\cdot) + b(t;\cdot,\cdot) + \kappa m(t;\cdot,\cdot)$$

is uniformly coercive in time. The associated dual problem is as follows.

Problem 3.5.2 (Dual problem)**.** *Given $\zeta \in H(t)$, find $\eta \in V(t)$ such that*

$$a_\kappa(t;\varphi,\eta) = m(t;\zeta,\varphi) \qquad \forall \varphi \in V(t).$$

By our boundedness assumption on the bilinear form $m(t;\cdot,\cdot)$, we deduce that the dual problem is well-posed. We further introduce the following regularity assumption on the solution $\eta \in V(t)$.

Assumption 3.5.13 (Regularity on dual problem)**.** *We assume the solution η to the dual problem belongs to the abstract space $\eta \in Z(t)$ and furthermore satisfies the estimate*

$$\|\eta\|_{Z(t)} \leq C\|\zeta\|_{H(t)} \tag{R}$$

for a constant $C > 0$ independent of t and ζ.

Provided all of the above listed assumptions are satisfied, we have the following optimal order error estimate derived in [35, Theorem 3.8].

Theorem 3.5.2 (Error estimate)**.** *Let us assume that the solution u to the mean-weak formulation has sufficiently smooth realisations and let $u_h = U_h^l$ denote the the lift of the semi-discrete problem with initial condition $u_{h,0} = U_{h,0}^l$. Then the following error bound holds*

$$\begin{aligned}\sup_{t\in[0,T]} \|u - u_h\|^2_{L^2(\Omega,\mathcal{H}(t))} &+ h^2 \int_0^T \|u - u_h\|_{L^2(\Omega,\mathcal{V}(t))} \\ &\leq \|u_0 - u_{h,0}\|^2_{L^2(\Omega,\mathcal{H}_0)} + Ch^4 \left(\sup_{t\in[0,T]} \|u\|^2_{L^2(\Omega,\mathcal{Z}(t))} + \int_0^T \|\partial^\bullet u\|^2_{L^2(\Omega,\mathcal{Z}(t))} \right).\end{aligned} \tag{3.5.13}$$

3.5.3 Monte-Carlo discretisation

We now proceed by combining the spatial semi-discretisation of the mean-weak formulation for the reformulated equation/system, with the Monte-Carlo method to estimate our quantity of interest, the mean solution of the pull-back $E[u]$. We first recall the Monte Carlo estimator $E_M[Y] : \otimes_{i=1}^M \Omega \to \mathcal{H}$ of the expectation of a Hilbert space-valued random variable $Y \in L^2(\Omega, \mathcal{H})$ is given by

$$E_M[Y] = \frac{1}{M} \sum_{i=1}^M \hat{Y}_i$$

where $M \in \mathbb{N}$ is the chosen number of samples taken and $\hat{Y}_i$ are independent identically distributed copies of the random variable Y and that we have the estimate

$$\|E[Y] - E_M[Y]\|_{L^2(\Omega^M,\mathcal{H})} \leq \frac{1}{\sqrt{M}}\|Y\|_{L^2(\Omega,\mathcal{H})}. \tag{3.5.14}$$

See Theorem 2.5.2 for further details. Thus if we consider the $L^2(\Omega^M, \mathcal{H})$-error between the mean solution $\mathbb{E}[u(t)]$ and the discrete solution $E_M[u_h(t)]$ at a given time $t \in [0, T]$, we have

$$\begin{aligned}
&\|E[u(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,\mathcal{H}(t))} \\
&\qquad \leq \|E[u(t)] - E[u_h(t)]\|_{L^2(\Omega^M,\mathcal{H}(t))} + \|E[u_h(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,\mathcal{H}(t))} \\
&\qquad \leq \|u(t) - u_h(t)\|_{L^2(\Omega,\mathcal{H}(t))} + \frac{1}{\sqrt{M}}\|u_h(t)\|_{L^2(\Omega,\mathcal{H}(t))} \\
&\qquad \lesssim \|u_0 - u_{h,0}\|_{L^2(\Omega,\mathcal{H}_0)} + h^2 + \frac{1}{\sqrt{M}},
\end{aligned}$$

where here we have used the stability estimate (3.5.9) on the semi-discrete solution. By a similar argument in the $L^2(\Omega^M, \mathcal{V}(t))$ we derive the following convergence rates.

Theorem 3.5.3. *Let us assume that the initial condition $u_{h,0} = U^l_{h,0}$ for the semi-discrete problem satisfies*

$$\|u_0 - u_{h,0}\|_{L^2(\Omega,\mathcal{H}_0} \leq ch^2.$$

Further, let us assume all of the conditions required in the semi-discrete error estimate hold, then we have the following error estimates

$$\sup_{t\in[0,T]} \|E[u(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,\mathcal{H}(t))} \lesssim h^2 + \frac{1}{\sqrt{M}} \tag{3.5.15}$$

$$\int_0^T \|E[u(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,\mathcal{V}(t))}\, dt \lesssim h + \frac{1}{\sqrt{M}}. \tag{3.5.16}$$

3.6 Discretisation of the reformulated advection-diffusion equation on the evolving reference surface

In this section, we apply the results from the previous abstract numerical analysis to a proposed finite element discretisation of the random advection-diffusion equation which arises after we employ the extended domain mapping method to reformulate our first model problem consisting of an advection-diffusion equation on a randomly evolving surface, onto the deterministically evolving reference surface, see Problem 3.4.2. We begin by first recapping some of the specific details for the reformulated problem. Note this application and the subsequent analysis which follows is motivated by and develops upon the results presented in [30, 35] which consider a specific deterministic case. We extend these results presented to our particular random geometric setting and prove in a similar fashion to [35] that the all the necessary assumptions of the abstract numerical analysis are indeed satisfied.

3.6.1 Summary of the reformulated advection-diffusion equation

The evolving reference domain for the extended domain mapping method was selected to be a compact smooth evolving surface $\Gamma_0(t) \subset \mathbb{R}^{n+1}$ whose evolution in time is described by the given normal velocity field $v_\nu^{\Gamma_0}$. We introduced an artificial advection of material points over the surface $\Gamma_0(t)$ by considering an ALE velocity field which we denoted by

$$v^{\Gamma_0} = v_\nu^{\Gamma_0} + v_{\tau,arb}^{\Gamma_0},$$

and for which we imposed suitable assumptions on the smoothness of the associated material flow, see Assumption 3.4.6. The initial advection-diffusion equation was then reformulated onto $\Gamma_0(t)$ with the prescribed stochastic domain mapping, and the resulting equation was to derive to be as follows

$$\begin{aligned}&\partial^\bullet_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega) + \sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega \nabla_{\Gamma_0(t)} \cdot v^{\Gamma_0} \\ &+ \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)}\hat{u}_\omega(v^{\Gamma_0}_{\tau,corr}(\omega) - v^{\Gamma_0}_{\tau,arb})) - \nabla_{\Gamma_0(t)} \cdot (\sqrt{g_{\Gamma_0}(\omega)} G^{-1}_{\Gamma_0}(\omega)\nabla_{\Gamma_0(t)}\hat{u}_\omega) = 0 \quad \text{on } \Gamma_0(t).\end{aligned}$$

A mean-weak formulation for the reformulated equation was then given as follows. Find $u \in W(V,H)$ such that for a.e.$t \in [0,T]$ we have

$$m(t; \partial^\bullet_{v^{\Gamma_0}} u, \varphi) + g(t, v^{\Gamma_0}; u, \varphi) + a(t; u, \varphi) + b(t; u, \varphi) = 0 \quad \forall \varphi \in V(t) \tag{3.6.1}$$

$$u(0) = u_0 \quad \text{in } H_0, \tag{3.6.2}$$

where $V(t) = L^2(\Omega, H^1(\Gamma_0(t)))$ and $H(t) = L^2(\Omega, L^2(\Gamma_0(t)))$, and the associated bilinear forms for the continuous problem are given by

$$\begin{aligned} m(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}}\varphi\psi \\ a(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \mathcal{A}\nabla_{\Gamma_0(t)}\varphi \cdot \nabla_{\Gamma_0(t)}\psi \\ b(t;\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \varphi\,\mathcal{B} \cdot \nabla_{\Gamma_0(t)}\psi \\ g(t,v^{\Gamma_0};\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \left(\partial^\bullet_{v^{\Gamma_0}}(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)} \cdot v^{\Gamma_0}\right)\varphi\psi \\ \tilde{a}(t,v^{\Gamma_0};\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \tilde{\mathcal{A}}(v^{\Gamma_0})\nabla_{\Gamma_0(t)}\varphi \cdot \nabla_{\Gamma_0(t)}\psi \\ \tilde{b}(t,v^{\Gamma_0};\varphi,\psi) &= \int_\Omega \int_{\Gamma_0(t)} \varphi\tilde{\mathcal{B}}(v^{\Gamma_0}) \cdot \nabla_{\Gamma_0(t)}\psi. \end{aligned}$$

For convenience, we adopt the following notation for the random coefficients

$$\begin{aligned}
\mathcal{A} &= \sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \\
\mathcal{B} &= \sqrt{g_{\Gamma_0}} (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0}) \\
\tilde{\mathcal{A}}(v) &= \partial_v^\bullet(\mathcal{A}) + \mathcal{A}(\nabla_{\Gamma_0(t)} \cdot v) - \nabla_{\Gamma_0(t)} v \mathcal{A} - \mathcal{A} \nabla_{\Gamma_0(t)} v^\top \\
\tilde{\mathcal{B}}(v) &= \partial_v^\bullet(\mathcal{B}) + \mathcal{B}(\nabla_{\Gamma_0(t)} \cdot v) - \nabla_{\Gamma_0(t)} v \mathcal{B}.
\end{aligned}$$

Furthermore, we will often refer to ALE velocity field by v and the associated strong material derivative by $\partial^\bullet$ provided there is no ambiguity to which velocity field is being considered. Suitable uniform estimates on the above random coefficients are derived in Lemma 3.4.1. We will now proceed by formulating our semi-discrete scheme for the reformulated equation, based upon the evolving surface finite element method first presented in [30]. However, contrary to the original method, we will choose the evolve the nodes of the discrete surface by the prescribed ALE velocity field v^{Γ_0} as follows.

3.6.2 Evolving discrete surface

Let us approximate the smooth evolving compact reference surface $\Gamma_0(t) \subset \mathbb{R}^3$, by an evolving polyhedral surface

$$\Gamma_h(t) = \bigcup_{T(t) \in \mathcal{T}_h(t)} T(t) \qquad \partial \Gamma_h(t) = \emptyset,$$

consisting of simplices $T(t) \in \mathcal{T}_h(t)$ whose maximum diameter is bounded uniformly in time by $h > 0$ and such that the triangulation $\mathcal{T}_h(t)$ is uniformly quasi-uniform in time, i.e. such that the in-ball radius $\sigma(T(t))$ of any simplex $T(t) \in \mathcal{T}_h(t)$ is uniformly bounded below by $\sigma(T(t)) \geq ch$, for a constant $c > 0$ independent of $t \in [0, T]$. We assume that the discrete surface in contained within a neighbourhood $\mathcal{N}(t)$ of $\Gamma(t)$ in which the Fermi coordinates

$$x = a(x,t) + d(x,t)\nu^{\Gamma_0}(x,t) \qquad \forall x \in \mathcal{N}(t)$$

are well-defined and we denote the induced curved simplices on $\Gamma_0(t)$ which are the images of simplices $T(t) \in \mathcal{T}_h(t)$ under the projective mapping $a(\cdot, t)$ by $T^l(t) = a(T(t), t)$. We further assume that there does not exists a double-covering under the projective mapping $a(\cdot, t)$, in the sense that its restriction onto the discrete surface at each $t \in [0, T]$

$$a(\cdot, t) : \Gamma_h(t) \to \Gamma_0(t) \tag{3.6.3}$$

forms a bijection between the two surfaces. We define a lift f^l of a given function $f \in C^0(\Gamma_h(t))$ onto the smooth surface by the relation

$$f^l(a(x,t), t) = f(x,t) \qquad x \in \Gamma_h(t) \tag{3.6.4}$$

and will denote the inverse lift of a given $f \in C^0(\Gamma_0(t))$ onto the discrete surface $\Gamma_h(t)$ by f^{-l}. We finally assume that the vertices $\{X_j(t)\}_{j=1}^N$ of the evolving discrete surface sit on the smooth surface $\Gamma_0(t)$, such that $\Gamma_h(t)$ forms an interpolant of $\Gamma_0(t)$ at each $t \in [0,T]$, and we will evolve the nodes of the triangulation by the given ALE velocity

$$X_j'(t) = v^{\Gamma_0}(X_j(t),t) \qquad \forall t \in [0,T].$$

This approach as previously discussed, has the added advantage of preserving the quality of the triangulation as the discrete surface evolves in time.

3.6.3 Evolving finite element spaces and the discrete material derivative

We next introduce a piece-wise linear finite element space on the evolving discrete surface and its corresponding lifted finite element space on the smooth surface $\Gamma_0(t)$ by

$$S_h(t) = \{\Phi_h \in C^0(\Gamma_h(t)) \,|\, \Phi_h|_{T(t)} \text{ is affine linear for all } T(t) \in \mathcal{T}_h(t)\} \tag{3.6.5}$$

$$S_h^l(t) = \{\varphi_h = \Phi_h^l \,|\, \Phi_h \in S_h(t)\}. \tag{3.6.6}$$

We observe by the smoothness of the projection mapping $a(\cdot,t) : \Gamma_h(t) \to \Gamma_0(t)$, that $S_h^l(t) \subset H^1(\Gamma_0(t))$. We denote the nodal basis of $S_h(t)$ by $\{\chi_j(\cdot,t)\}_{j=1}^N$, that is the piece-wise linear functions on $\Gamma_h(t)$ determined by the relations

$$\chi_j^t(X_k(t),t) = \delta_{k,j} \qquad \forall k = 1,...,N, \tag{3.6.7}$$

and we define a discrete material velocity field V_h on the evolving discrete surface $\Gamma_h(t)$ by the interpolant of the smooth ALE velocity field v^{Γ_0},

$$V_h(x,t) = \sum_{j=1}^N X_j'(t)\,\chi_j(x,t) \qquad \forall x \in \Gamma_h(t). \tag{3.6.8}$$

This induces a material flow of points on the smooth surface $\Gamma_0(t)$, under the lifting mapping (3.6.3). More precisely, if we define a trajectory $\{Y(t)\}_{t\in[0,T]}$ on $\Gamma_0(t)$ by

$$Y(t) = a(X(t),t)$$

with the path $X(t)$ evolving by the discrete material velocity $X'(t) = V_h(X(t),t)$, then the velocity of $Y(t)$ may be computed as

$$Y'(t) = \nabla a(X(t),t) \cdot X'(t) + a_t(X(t),t).$$

Consequently, the material velocity field v_h on $\Gamma_0(t)$ associated to the discrete material velocity V_h is defined by

$$v_h(a(x,t),t) = (\mathcal{P}_{\Gamma_0}(x,t) - d(x,t)\mathcal{H}^{\Gamma_0}(x,t))V_h(x,t) - d_t(x,t)\nu^{\Gamma_0}(x,t) - d(x,t)\nu_t^{\Gamma_0}(x,t) \quad x \in \Gamma_h(t).$$

It is worth noting here that the material velocity field v_h, is not given by the interpolant of the ALE velocity field in the lifted finite element space $V_h^l(t)$. We may now define the discrete material derivative on the evolving discrete surface with respect to the discrete velocity V_h, and the discrete material derivative on the smooth surface with respect to v_h, element-wise by

$$\partial_h^\bullet \Phi_h|_{T(t)} = \big(\Phi_{h,t} + \nabla_{\Gamma_h(t)}\Phi_h \cdot V_h\big)\,|_{T(t)} \tag{3.6.9}$$

$$\partial_h^\bullet \varphi_h|_{T^l(t)} = \big(\varphi_{h,t} + \nabla_{\Gamma_0(t)}\varphi_h \cdot v_h\big)\,|_{T^l(t)}, \tag{3.6.10}$$

where we recall that $T^l(t) = T(t)^l$ denotes the lifted curved simplex on $\Gamma_0(t)$.

3.6.4 The semi-discrete problem

Within the context of the abstract framework, the two abstract norms which equip the evolving finite element space $\mathcal{V}_h(t) = S_h(t)$, will in our particular application be given by are given by

$$\|\Phi\|_{\mathcal{H}_h(t)} = \|\Phi_h\|_{L^2(\Gamma_h(t))} \qquad \|\Phi\|_{\mathcal{V}_h(t)} = \|\Phi_h\|_{H^1(\Gamma_h(t))}$$

and naturally satisfy the requirement $\|\Phi_h\|_{\mathcal{H}_h(t)} \leq \|\Phi_h\|_{\mathcal{V}_h(t)}$. The random discrete function spaces are given by

$$H_h(t) = L^2(\Omega, L^2(\Gamma_h(t))) \qquad V_h(t) = L^2(\Omega, H^1(\Gamma_h(t))),$$

and the abstract push-forward operator $\phi_t : \mathcal{V}_h(0) \to \mathcal{V}_h(t)$ describing the evolution of the finite element space, is given by the flow map associated to the discrete material velocity V_h, i.e.

$$\phi_t(\Phi_h)(G_h(x,t),t) = \Phi_h(x,t) \qquad x \in \Gamma_h(t) \qquad \text{for } \Phi_h \in S_h(0),$$

where $G_h(\cdot,t) : \Gamma_h(0) \to \Gamma_h(t)$ satisfies

$$G_h'(x,t) = V_h(G_h(x,t),t).$$

The discrete bilinear form approximating their continuous counterparts on the discrete surface are defined for $\Phi_h, \Psi_h \in H_h(t), V_h(t)$ by

$$\begin{aligned}
m_h(t;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \sqrt{g_{\Gamma_0}^{-l}}\,\Phi_h \Psi_h \\
a_h(t;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \mathcal{A}^{-l} \nabla_{\Gamma_h(t)}\Phi_h \cdot \nabla_{\Gamma_h(t)}\Psi_h \\
b_h(t;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \Phi_h \mathcal{B}^{-l} \cdot \nabla_{\Gamma_h(t)}\Psi_h \\
g_h(t,V_h;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \left(\partial_h^\bullet\left(\sqrt{g_{\Gamma_0}^{-l}}\right) + \sqrt{g_{\Gamma_0}^{-l}}\,\nabla_{\Gamma_h(t)} \cdot V_h\right)\Phi_h \Psi_h,
\end{aligned}$$

and the semi-discrete problem for the reformulated adveciton-diffusion equation is as follows.

Problem 3.6.1 (Semi-discrete problem). *Given $U_{h,0} \in H_h(0)$, find $U_h \in \tilde{C}^1_{V_h}$ such that for a.e. $t \in [0,T]$,*

$$m_h(t; \partial_h^\bullet U_h, \Phi_h) + g_h(t, V_h; U_h, \Phi_h) + a_h(t; U_h, \Phi_h) + b_h(t; U_h, \Phi_h) = 0 \tag{3.6.11}$$

for all $\Phi_h \in V_h(t)$ and which furthermore satisfies the initial condition

$$U_h(0) = U_{h,0} \quad \textit{in } H_h(0).$$

We next verify that the discrete bilinear forms satisfy all the necessary assumptions required in the abstract analysis to derive the existence and uniqueness of a solution to the semi-discrete problem. We first note that the assumptions $(M_h 1), (A_h 1), (A_h 2)$ and $(B_h 1)$ will immediately follow from the derived uniform estimates on the random coefficients established in Lemma 3.4.1. For the discrete transport properties $(T_h 1), (T_h, 2), (T_h 3)$, we require the following result, which follows by applying the Leibniz transport formula element-wise, see [30] for details.

Lemma 3.6.1 (Leibniz discrete transport property). *Given a sufficiently smooth function f defined over the evolving discrete surface $\Gamma_h(t)$, we have*

$$\frac{d}{dt}\int_{\Gamma_h(t)} f \, dA_h = \int_{\Gamma_h(t)} \partial_h f + f \nabla_{\Gamma_h(t)} \cdot V_h \, dA_h.$$

This leads to the following transport properties for the discrete bilinear forms

$$\begin{aligned}
\frac{d}{dt} m_h(t; \Phi_h, \Psi_h) &= m_h(t; \partial_h^\bullet \Phi_h, \Psi_h) + m_h(t; \Phi_h, \partial_h^\bullet \Psi_h) + g_h(t, V_h; \Phi_h, \Psi_h) \\
\frac{d}{dt} a_h(t; \Phi_h, \Psi_h) &= a_h(t; \partial_h^\bullet \Phi_h, \Psi_h) + a_h(t; \Phi_h, \partial_h^\bullet \Psi_h) + \tilde{a}_h(t, V_h; \Phi_h, \Psi_h) \\
\frac{d}{dt} b_h(t; \Phi_h, \Psi_h) &= b_h(t; \partial_h^\bullet \Phi_h, \Psi_h) + b_h(t; \Phi_h, \partial_h^\bullet \Psi_h) + \tilde{b}_h(t, V_h; \Phi_h, \Psi_h),
\end{aligned}$$

where the additional bilinear forms are given by

$$\begin{aligned}
\tilde{a}(t, V_h; \Phi_h, \Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \tilde{\mathcal{A}}_h(V_h) \nabla_{\Gamma_h(t)} \Phi_h \cdot \nabla_{\Gamma_h(t)} \Psi_h \\
\tilde{b}(t, V_h; \Phi_h, \Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \Phi_h \, \tilde{\mathcal{B}}_h(V_h) \cdot \nabla_{\Gamma_h(t)} \Psi_h
\end{aligned}$$

for the given random coefficients

$$\begin{aligned}
\tilde{\mathcal{A}}_h(V_h) &= \partial_h^\bullet \left(\mathcal{A}^{-l}\right) + (\nabla_{\Gamma_h(t)} \cdot V_h)\mathcal{A}^{-l} - \nabla_{\Gamma_h(t)} V_h \mathcal{A}^{-l} - \mathcal{A}^{-l} \nabla_{\Gamma_h(t)} V_h^\top \\
\tilde{\mathcal{B}}_h(V_h) &= \partial_h^\bullet \left(\mathcal{B}^{-l}\right) + (\nabla_{\Gamma_h(t)} \cdot V_h)\mathcal{B}^{-l} - \nabla_{\Gamma_h(t)} V_h \mathcal{B}^{-l},
\end{aligned}$$

where we recall

$$\mathcal{A}(\omega) = \sqrt{g_{\Gamma_0}(\omega)} G_{\Gamma_0}^{-1}(\omega) \qquad \mathcal{B}(\omega) = \sqrt{g_{\Gamma_0(\omega)}} (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0}(\omega)).$$

By the uniform estimates on the material derivative of the random coefficients derived in Lemma 3.4.1, and soon to follow estimates on the lifting mapping, we deduce the following assumptions $(M_h2), (A_h3)$ and (B_h2) are satisfied, and consequently deduce the existence and uniqueness of a solution to the semi-discrete problem by Theorem 3.5.1. We conclude this section, before continuing onto verifying all the assumptions for of the abstract error analysis hold, by deriving the following transport properties on the smooth surface $\Gamma_0(t)$ with respect to the velocity field v_h,

$$\begin{aligned}
\frac{d}{dt}m(t;\varphi,\psi) &= m(t;\partial_h^\bullet\varphi,\psi) + m(t;\varphi,\partial_h^\bullet\psi) + g(t,v_h;\varphi_h,\psi)\\
\frac{d}{dt}a(t;\varphi,\psi) &= a(t;\partial_h^\bullet\varphi,\psi) + a(t;\varphi,\partial_h^\bullet\psi) + \tilde{a}(t,v_h;\varphi,\psi)\\
\frac{d}{dt}b(t;\varphi,\psi) &= b(t;\partial_h^\bullet\varphi,\psi) + b(t;\varphi,\partial_h^\bullet\psi) + \tilde{b}_h(t,v_h;\varphi,\psi),
\end{aligned}$$

Here the additional bilinear forms defined over the smooth surface are given by

$$\begin{aligned}
\tilde{a}(t,v_h;\varphi,\psi) &= \int_\Omega\int_{\Gamma_0(t)} \tilde{\mathcal{A}}(v_h)\nabla_{\Gamma_0(t)}\varphi\cdot\nabla_{\Gamma_0(t)}\psi\\
\tilde{b}(t,v_h;\varphi,\psi) &= \int_\Omega\int_{\Gamma_0(t)} \varphi\,\tilde{\mathcal{B}}(v_h)\cdot\nabla_{\Gamma_0(t)}\psi
\end{aligned}$$

where the random coefficients are similarly given by

$$\begin{aligned}
\tilde{\mathcal{A}}_h(v_h) &= \partial_h^\bullet\mathcal{A} + (\nabla_{\Gamma_0(t)}\cdot v_h)\mathcal{A} - \nabla_{\Gamma_0(t)}v_h\mathcal{A} - \mathcal{A}\nabla_{\Gamma_0(t)}v_h^\top\\
\tilde{\mathcal{B}}_h(v_h) &= \partial_h^\bullet\mathcal{B} + (\nabla_{\Gamma_0(t)}\cdot v_h)\,\mathcal{B} - \nabla_{\Gamma_0(t)}v_h\mathcal{B}.
\end{aligned}$$

3.6.5 Error analysis and a convergence result

We now verify that all the stated assumptions given in the abstract numerical analysis hold, and thus deduce an optimal order error estimate for our proposed semi-discretisation of the continuous mean-weak formulation. This will follow a similar fashion to the application presented in [35], but where we extend and generalise the results of [30, 35] to our random setting. In the abstract framework, the deterministic Hilbert space $\mathcal{Z}(t)$ representing a function space contained within $\mathcal{V}(t) = H^1(\Gamma_0(t))$ consisting of functions of higher regularity, will be given for our particular formulation by

$$\mathcal{Z}(t) = H^2(\Gamma_0(t)).$$

Furthermore, the abstract space $\mathcal{Z}_{0,h}(t)$ and $\mathcal{Z}_0(t)$ for which there exists an abstract lifting mapping $\Lambda_h(\cdot,t) : \mathcal{Z}_{0,h}(t) \to \mathcal{Z}_0(t)$, are given by

$$\mathcal{Z}_{0,h}(t) = C^0(\Gamma_h(t)) \qquad \mathcal{Z}_0(t) = C^0(\Gamma_0(t))$$

and the abstract mapping defined by the lifting mapping (3.6.4)

$$\Lambda_h(\Phi_h, t) = \Phi_h^l \qquad \text{for } \Phi_h \in S_h(t).$$

The following estimates on the lifting mapping are derived in [42, Lemma 4.2] from which we deduce that the assumptions $(L1)$ and $(L2)$ are satisfied.

Lemma 3.6.2 (Lifting assumption)**.** *There exists constants $C_1, C_2 > 0$ independent of $h \in (0, h_0)$ and of $t \in [0, T]$, such that for any $\Phi_h \in S_h$ with lift $\varphi_h \in S_h^l$, we have*

$$C_1 \|\Phi_h\|_{L^2(\Gamma_h(t))} \leq \|\varphi_h\|_{L^2(\Gamma_0(t))} \leq C_2 \|\Phi_h\|_{L^2(\Gamma_h(t))},$$
$$C_1 \|\nabla_{\Gamma_h(t)} \Phi_h\|_{L^2(\Gamma_h(t))} \leq \|\nabla_{\Gamma_0(t)} \varphi_h\|_{L^2(\Gamma_0(t))} \leq C_2 \|\nabla_{\Gamma_h(t)} \Phi_h\|_{L^2(\Gamma_h(t))}.$$

We next note that the push-forward operator $\phi_t^l : S_h^l(0) \to S_h^l(t)$ defined by the lift onto the smooth surface $\Gamma_0(t)$ under the projection operator $a(\cdot, t)$, of the flow map associated to the discrete material velocity V_h, is precisely the flow associated with the discrete material velocity field v_h on $\Gamma_0(t)$. Consequently, the push-forward operator ϕ_t^l may be extended to the spaces

$$\Phi_t^l : L^2(\Gamma_0(0)) \to L^2(\Gamma_0(t))$$

and forms compatible pairs $(\mathcal{H}, \phi_l^t)$ and $(\mathcal{V}, \phi_l^t|_{\mathcal{V}(0)})$, by the assumed smoothness of the initial ALE velocity field. In particular, the strong material derivative associated to the push-forward operator ϕ_l^t is given by the discrete material derivative $\partial_{v_h}^\bullet$ on the smooth surface $\Gamma_0(t)$. We next state an interpolation estimate on the space $\mathcal{Z}_0(t)$ given in [31, Lemma 5.3]. We recall that the interpolation operator $I_h : H^2(\Gamma_0(t)) \to S_h^l(t)$, is defined by lifting the Lagrangian interpolant on the discrete surface onto the smooth domain.

Lemma 3.6.3 (Interpolation estimate)**.** *Given any $\eta \in H^2(\Gamma_0(t))$, there exists a constant $c > 0$ independent of h and $t \in [0, T]$ such that*

$$\|\eta - I_h \eta\|_{L^2(\Gamma_0(t))} + h \|\nabla_{\Gamma_0(t)}(\eta - I_h \eta)\|_{L^2(\Gamma_0(t))} \leq ch^2 \|\eta\|_{H^2(\Gamma_0(t))}. \tag{3.6.12}$$

To derive the assumed bounds on the geometric pertubation of the continuous bilinear forms, we will first require some preliminary estimates on the order of approximation of the geometry. We begin by first introducing some notation, relating the geometric error arising from the approximation of the bilinear forms on the discrete surface. We recall, that the lift of a function f defined over the discrete surface $\Gamma_h(t)$, onto the smooth surface is given by $f^l(a(x, t), t) = f(x, t)$, $x \in \Gamma_h(t)$ and therefore by the chain rule we have, where we shall suppress the time-dependency in the notation for convenience,

$$\nabla_{\Gamma_h(t)} f(x) = P_h(x)(I - d(x)\mathcal{H}(x)) \nabla_{\Gamma_0(t)} f^l(a(x)),$$

with $P_h = I - \nu_h \otimes \nu_h$ denoting the discrete projection operator defined element-wise. Consequently, lifting the discrete bilinear form $b_h(t; \cdot, \cdot)$ onto the smooth surface, for any $\Phi_h, \Psi_h \in$

$L^2(\Omega, S_h(t))$ with respective lifts φ_h, ψ_h, we have

$$
\begin{aligned}
b_h(t;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \Phi_h \mathcal{B}^{-l} \cdot \nabla_{\Gamma_h(t)} \Psi_h \\
&= \int_\Omega \int_{\Gamma_0(t)} \frac{1}{\delta_h^l} \varphi_h \mathcal{B} \cdot P_h(I - d\mathcal{H}) \nabla_{\Gamma_0(t)} \varphi, \qquad (3.6.13)
\end{aligned}
$$

with δ_h denoting the quotient between the discrete and smooth surface measures, $\delta_h dA_h = dA$. Similarly, we have

$$
\begin{aligned}
a_h(t;\Phi_h,\Psi_h) &= \int_\Omega \int_{\Gamma_h(t)} \mathcal{A}^{-l} \nabla_{\Gamma_h(t)} \Phi_h \cdot \nabla_{\Gamma_h(t)} \Psi_h \\
&= \int_\Omega \int_{\Gamma_h(t)} \frac{1}{\delta_h^l} P(I - d\mathcal{H}) P_h \mathcal{A} P_h (I - d\mathcal{H}) P \nabla_{\Gamma_0(t)} \varphi_h \cdot \nabla_{\Gamma_0(t)} \psi_h. \qquad (3.6.14)
\end{aligned}
$$

We thus define the terms

$$
\begin{aligned}
R_h^1 &= \frac{1}{\delta_h} \mathcal{A}^{-1} P(I - d\mathcal{H}) P_h \mathcal{A} P_h (I - d\mathcal{H}) P \qquad (3.6.15)\\
R_h^2 &= \frac{1}{\delta_h} P P_h (I - d\mathcal{H}) P, \qquad (3.6.16)
\end{aligned}
$$

and have the following estimates based on [31, Lemma 5.1].

Lemma 3.6.4 (Geometric error bounds)**.** *Assuming $\Gamma_0(t)$ and $\Gamma_h(t)$ are as above, then we have the estimates*

$$
\begin{aligned}
\|d\|_{L^\infty(\Gamma_h(t))} &\le ch^2 \qquad (3.6.17)\\
\|1 - \delta_h\|_{L^\infty(\Gamma_h(t))} &\le ch^2 \qquad (3.6.18)\\
\|P(I - R_h^2)\|_{L^\infty(\Gamma_h(t))} &\le ch^2 \qquad (3.6.19)\\
\|(I - R_h^1(\omega))\mathcal{P}\|_{L^\infty(\Gamma_h(t))} &\le ch^2. \qquad (3.6.20)
\end{aligned}
$$

Proof. The estimates (3.6.17) and (3.6.18) have already been established in [31, Lemma 5.1]. For (3.6.19), we apply the previous estimates to deduce

$$
P(I - R_h^2) = P(I - \frac{1}{\delta_h} P_h (I - d\mathcal{H}) P) = P - P P_h P + \mathcal{O}(h^2) = -P\nu_h \otimes P\nu_h + \mathcal{O}(h^2).
$$

With the estimate $|P\nu_h| \le ch$ given in [42, Lemma 4.1], we obtain (3.6.19). □

Furthermore, we require the following estimates on the material derivative of geometric quantities, which has been previously derived in [31, Lemma 5.4].

Lemma 3.6.5 (Discrete material derivative estimates)**.** *There exists constants $c > 0$ indepen-*

dent of $h \in (0, h_0)$ and $t \in [0,T]$ such that we have the following estimates

$$\sup_{t\in(0,T)} \|\partial_h^\bullet d\|_{L^\infty(\Gamma_h(t))} \le ch^2 \tag{3.6.21}$$

$$\sup_{t\in(0,T)} \|\partial_h^\bullet (P_h\nu)\|_{L^\infty(\Gamma_h(t))} \le ch^2 \tag{3.6.22}$$

$$\sup_{t\in(0,T)} \|\partial_h^\bullet \delta_h\|_{L^\infty(\Gamma_h(t))} \le ch^2. \tag{3.6.23}$$

We may now bound the geometric pertubation errors as follows, where we have adapted the proofs presented in [35, 42] to our random setting.

Lemma 3.6.6. *Given any $\Phi_h, \Psi_h \in L^2(\Omega, S_h(t))$ with respective lifts φ_h, ψ_h, we have*

$$\begin{aligned}
|m(t;\varphi_h,\psi_h) - m_h(t;\Phi_h,\Psi_h)| &\le ch^2\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\\
|a(t;\varphi_h,\psi_h) - a_h(t;\Phi_h,\Psi_h)| &\le ch^2\|\varphi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))}\\
|b(t;\varphi_h,\psi_h) - b_h(t;\Phi_h,\Psi_h)| &\le ch^2\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))}.
\end{aligned}$$

Furthermore, we have the estimates

$$|g(t,v_h;\varphi_h,\psi_h) - g_h(t,V_h;\Phi_h,\Psi_h)| \le ch^2\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))} \tag{3.6.24}$$

$$|\tilde{a}(t,v_h;\varphi_h,\psi_h) - \tilde{a}_h(t,V_h;\Phi_h,\Psi_h)| \le ch^2\|\varphi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))} \tag{3.6.25}$$

$$|\tilde{b}(t,v_h;\varphi_h,\psi_h) - \tilde{b}_h(t,V_h;\Phi_h,\Psi_h)| \le ch^2\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\psi_h\|_{L^2(\Omega,H^1(\Gamma_0(t)))}. \tag{3.6.26}$$

Proof. For the first estimate, we observe by lifting the discrete integral onto the smooth surface that

$$m(t;\varphi_h,\psi_h) - m_h(t;\Phi_h,\Psi_h) = \int_\Omega \int_{\Gamma_0(t)} \left(1 - \frac{1}{\delta_h^l}\right)\varphi_h\psi_h\sqrt{g_{\Gamma_0}}.$$

Consequently, the estimate follows from the uniform bound on the random coefficient $\sqrt{g_{\Gamma_0}}$ given in Lemma 3.4.1 and the previous geometric estimate (3.6.18). Similarly, we observe from (3.6.14) and (3.6.13) that

$$a(t;\varphi_h,\psi_h) - a_h(t;\Phi_h,\Psi_h) = \int_\Omega \int_{\Gamma_0(t)} \mathcal{A}P\left(I - R_h^1\right)\nabla_{\Gamma_0(t)}\varphi_h \cdot \psi_h \tag{3.6.27}$$

$$b(t;\varphi_h,\psi_h) - b_h(t;\Phi_h,\Psi_h) = \int_\Omega \int_{\Gamma_0(t)} \varphi_h\,\mathcal{B}\cdot P\left(I - R_h^2\right)\nabla_{\Gamma_0(t)}\psi, \tag{3.6.28}$$

where we note in particular that $\mathcal{B} = \sqrt{g_{\Gamma_0}}(v^{\Gamma_0}_{\tau,arb} - v^{\Gamma_0}_{\tau,corr})$ is tangential to the surface $\Gamma_0(t)$. This therefore leads to the stated estimates by the given geometric estimates (3.6.19) and (3.6.20) as well as the uniform bounds on the random coefficients. It remains to show the bounds on the bilinear forms which arise from the transport properties. Let us start with the estimate

(3.6.24). Lifting the discrete bilinear form $g_h(t;\cdot,\cdot)$ onto the smooth surface yields

$$g_h(t;\Phi_h,\Psi_h) = \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \varphi_h \psi_h \frac{1}{\delta_h^l}. \tag{3.6.29}$$

Differentiating in time, we may apply the transport property on the smooth surface $\Gamma_0(t)$ with respect to the discrete velocity v_h to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \varphi_h \psi_h \frac{1}{\delta_h^l} \\
&\quad = \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \partial_h^\bullet \varphi_h \psi_h \frac{1}{\delta_h^l} + \sqrt{g_{\Gamma_0}} \varphi_h \partial_h^\bullet \psi_h \frac{1}{\delta_h^l} + \int_\Omega \int_{\Gamma_0(t)} \frac{1}{\delta_h^l} \left(\partial_h^\bullet \left(\sqrt{g_{\Gamma_0}}\right) + \sqrt{g_{\Gamma_0}} \nabla_{\Gamma_0(t)} \cdot v_h\right) \varphi_h \psi_h \\
&\quad + \int_\Omega \int_{\Gamma_0(t)} \partial_h^\bullet \left(\frac{1}{\delta_h^l}\right) \sqrt{g_{\Gamma_0}} \varphi_h \psi_h
\end{aligned}$$

Pushing the first integral back onto the discrete surface gives

$$\begin{aligned}
&= m_h(t;\partial_h^\bullet \Phi_h,\Psi_h) + m_h(t;\Phi_h,\partial_h^\bullet \Psi_h) + g(t,v_h;\varphi_h,\psi_h) \\
&- \int_\Omega \int_{\Gamma_0(t)} \left(1 - \frac{1}{\delta_h^l}\right) \left(\partial_h^\bullet(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}} \nabla_{\Gamma_0(t)} \cdot v_h\right) \varphi_h \psi_h + \int_\Omega \int_{\Gamma_0(t)} \partial_h^\bullet \left(\frac{1}{\delta_h^l}\right) \sqrt{g_{\Gamma_0}} \varphi_h \psi_h.
\end{aligned}$$

Comparing the above equation with the discrete transport property

$$\frac{d}{dt} m_h(t;\Phi_h,\Psi) = m_h(t;\partial_h^\bullet \Phi_h,\Psi) + m_h(t;\Phi_h,\partial_h^\bullet \Psi) + g_h(t,V_h;\Phi_h,\Psi_h)$$

leads to the estimate

$$\begin{aligned}
&|g(t,v_h;\varphi_h,\psi_h) - g_h(t,V_h;\Phi_h,\Psi_h)| \\
&\qquad \le c\|1 - \frac{1}{\delta_h^l}\|_{L^\infty(\Gamma_0(t))} \|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))} \|\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))} \\
&\qquad + c\|\partial_h^\bullet(\frac{1}{\delta_h^l})\|_{L^\infty(\Gamma_0(t))} \|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))} \|\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))},
\end{aligned}$$

where we have again used the derived uniform bounds established in Lemma 3.4.1 on the random coefficients and their material derivatives. We may employ the given geometric estimates to obtain the desire bound. We proceed in a similar manner for the estimate (3.6.25). We first lift the discrete bilinear form $a_h(t;\cdot,\cdot)$ onto the smooth surface as previously derived in (3.6.14), to obtain

$$a_h(t;\Phi_h,\Psi_h) = \int_\Omega \int_{\Gamma_0(t)} \mathcal{A} R_h^1 \nabla_{\Gamma_0(t)} \varphi_h \cdot \nabla_{\Gamma_0(t)} \psi_h$$

and then take the time derivative of the lifted integral to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\Gamma_0(t)} \mathcal{A}R_h^1\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h\\
&\quad=\int_\Omega\int_{\Gamma_0(t)} \mathcal{A}R_h^1\nabla_{\Gamma_0(t)}\partial_h^\bullet\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h+\mathcal{A}R_h^1\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\partial_h^\bullet\psi_h\\
&\quad+\int_\Omega\int_{\Gamma_0(t)}\left(\partial_h^\bullet\mathcal{A}+(\nabla_{\Gamma_0(t)}\cdot v_h)\mathcal{A}-\nabla_{\Gamma_0(t)}v_h\mathcal{A}\right)R_h^1\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h\\
&\quad-\int_\Omega\int_{\Gamma_0(t)}\mathcal{A}R_h^1\nabla_{\Gamma_0(t)}v_h^\top\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h+\int_\Omega\int_{\Gamma_0(t)}\mathcal{A}\partial_h^\bullet\left(R_h^1\right)\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h.
\end{aligned}
$$

The first term pulled-back onto the reference gives

$$
\begin{aligned}
&=a_h(t;\partial_h^\bullet\Phi_h,\Psi_h)+a_h(t;\Phi_h,\partial_h^\bullet\psi_h)+\tilde{a}(t,v_h;\varphi_h,\psi_h)\\
&-\int_\Omega\int_{\Gamma_0(t)}\left(\partial_h^\bullet\mathcal{A}+(\nabla_{\Gamma_0(t)}\cdot v_h)\mathcal{A}-\nabla_{\Gamma_0(t)}v_h\mathcal{A}\right)\left(P-R_h^1\right)\nabla_{\Gamma_0(t)}\varphi\cdot\nabla_{\Gamma_0(t)}\psi_h\\
&+\int_\Omega\int_{\Gamma_0(t)}\mathcal{A}(P-R_h^1)\nabla_{\Gamma_0(t)}v_h^\top\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h+\int_\Omega\int_{\Gamma_0(t)}\mathcal{A}\partial_h^\bullet\left(R_h^1\right)\nabla_{\Gamma_0(t)}\varphi_h\cdot\nabla_{\Gamma_0(t)}\psi_h,
\end{aligned}
$$

which leads to the estimate

$$
\begin{aligned}
&|\tilde{a}(t,v_h;\varphi_h,\psi_h)-\tilde{a}_h(t,V_h;\Phi_h,\Psi_h)|\\
&\qquad\le c\|P-R_h^1\|_{L^\infty(\Gamma_0(t))}\|\nabla_{\Gamma_0(t)}\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\nabla_{\Gamma_0(t)}\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\\
&\qquad+\|\partial_h^\bullet(R_h^1)\|_{L^\infty(\Gamma_0(t))}\|\nabla_{\Gamma_0(t)}\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\nabla_{\Gamma_0(t)}\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}.
\end{aligned}
$$

Recall that

$$
R_h^1=\frac{1}{\delta_h^l}\mathcal{A}^{-1}P(I-d\mathcal{H})P_h\mathcal{A}P_h(I-d\mathcal{H})P,
$$

therefore with the geometric estimates given in Lemma 3.6.5 we deduce

$$
\begin{aligned}
\partial_h^\bullet(R_h^1)&=\partial_h^\bullet\left(\mathcal{A}^{-1}PP_h\mathcal{A}P_hP\right)+\mathcal{O}(h^2)\\
&=\partial_h^\bullet\left(\mathcal{A}^{-1}(I-P\nu_h\otimes\nu_h)\mathcal{A}(I-\nu_h\otimes P\nu_h)\right)+\mathcal{O}(h^2)\\
&=\partial_h^\bullet(\mathcal{A}^{-1}\mathcal{A})+\mathcal{O}(h^2)=\mathcal{O}(h^2),
\end{aligned}
$$

and therefore obtain the desired estimate. We conclude by deriving last estimate for the discrete bilinear form $\tilde{b}_h(t,v_h;\cdot,\cdot)$. We employ the same argument and lift $b_h(t;\cdot,\cdot)$ onto the smooth surface

$$
b_h(t;\Phi_h,\Psi_h)=\int_\Omega\int_{\Gamma_0(t)}\varphi_h\,\mathcal{B}\cdot R_h^2\nabla_{\Gamma_0(t)}\psi_h \tag{3.6.30}
$$

and then differentiate in time to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{\Gamma_0(t)} \varphi_h\,\mathcal{B}\cdot R_h^2\nabla_{\Gamma_0(t)}\psi_h\\
&\quad=\int_\Omega\int_{\Gamma_0(t)} \partial_h^\bullet\varphi_h\,\mathcal{B}\cdot R_h^2\nabla_{\Gamma_0(t)}\psi_h+\varphi_h\,\mathcal{B}\cdot R_h^2\nabla_{\Gamma_0(t)}\partial_h^\bullet\psi_h\\
&\quad+\int_\Omega\int_{\Gamma_0(t)} \varphi_h\,\left((\nabla_{\Gamma_0(t)}\cdot v_h)\mathcal{B}+\partial_h^\bullet\mathcal{B}\right)\cdot R_h^2\nabla_{\Gamma_0(t)}\psi_h+\int_\Omega\int_{\Gamma_0(t)} \varphi_h\mathcal{B}\cdot R_h^2\nabla_{\Gamma_0(t)}v_h^\top\nabla_{\Gamma_0(t)}\psi_h\\
&\quad+\int_\Omega\int_{\Gamma_0(t)} \varphi_h\mathcal{B}\cdot\partial_h^\bullet\left(R_h^2\right)\nabla_{\Gamma_0(t)}\psi_h.
\end{aligned}$$

This equates to

$$\begin{aligned}
&= b_h(t;\partial_h^\bullet\Phi_h,\Psi_h)+b_h(t;\Phi_h,\partial_h^\bullet\Psi_h)+\tilde{b}(t,v_h;\varphi_h,\psi_h)\\
&-\int_\Omega\int_{\Gamma_0(t)} \varphi_h\,\left((\nabla_{\Gamma_0(t)}\cdot v_h)\mathcal{B}+\partial_h^\bullet\mathcal{B}\right)\cdot(P-R_h^2)\nabla_{\Gamma_0(t)}\psi_h\\
&-\int_\Omega\int_{\Gamma_0(t)} \varphi_h\mathcal{B}\cdot(P-R_h^2)\nabla_{\Gamma_0(t)}v_h^\top\nabla_{\Gamma_0(t)}\psi_h+\int_\Omega\int_{\Gamma_0(t)} \varphi_h\mathcal{B}\cdot\partial_h^\bullet\left(R_h^2\right)\nabla_{\Gamma_0(t)}\psi_h
\end{aligned}$$

and hence gives

$$\begin{aligned}
&|\tilde{b}(t,v_h;\varphi_h,\psi_h)-\tilde{b}_h(t,V_h;\Phi_h,\Psi_h)|\\
&\qquad\le c\|P-R_h^2\|_{L^\infty(\Gamma_0(t))}\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\nabla_{\Gamma_0(t)}\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\\
&\qquad+c\|\partial_h^\bullet(R_h^2)\|_{L^\infty(\Gamma_0(t))}\|\varphi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}\|\nabla_{\Gamma_0(t)}\psi_h\|_{L^2(\Omega,L^2(\Gamma_0(t)))}
\end{aligned}$$

and thus the required estimate.

□

It therefore follows that all the assumptions $(G1)-(G9)$ on the geometric pertubation of the continuous bilinears forms hold. We next continue by considering the error induced by the approximation of the smooth velocity field. We first begin with a preliminary result derived in [31, Lemma 5.6], which bounds the difference between the smooth ALE velocity field v and the discrete material velocity field on the smooth surface $\Gamma_0(t)$.

Lemma 3.6.7. *Let v_h denote the discrete material velocity field on $\Gamma_0(t)$ and v the smooth ALE velocity. Then, there exists a constant $C>0$ such that we have the following estimate*

$$\|v-v_h\|_{L^\infty(\Gamma_0(t))}+h\|\nabla_{\Gamma_0(t)}(v-v_h)\|_{L^\infty(\Gamma_0(t))}\le ch^2. \tag{3.6.31}$$

Proof. We provide a brief account of how this estimate is derived. We first recall that the discrete material velocity field on the smooth surface $\Gamma_0(t)$ was given by

$$v_h(a(x,t),t)=-d_t(x,t)\nu(x,t)-\nu_t(x,t)d(x,t)+(I-d(x,t)\mathcal{H}(x,t))P(x,t)V_h(x,t)\qquad x\in\Gamma_h(t).$$

Observing that $v_\nu=-d_t\nu$ and the discrete material velocity on the discrete surface is defined as $V_h = I_h v$, we deduce where we shall suppress parameters for convenience, that

$$\begin{aligned}(v - v_h)(a) &= (v - v_\nu)(a) - PI_h v + d(\nu_t + \mathcal{H}PI_h v(a)) \\ &= P(v - I_h v)(a) + d(\nu_t + \mathcal{H}PI_h v(a))\end{aligned}$$

thus obtaining the first estimate and where the second estimate will follow by differentiating the above expression. □

This leads to the following estimates.

Lemma 3.6.8 (Velocity field estimates)**.** *There exists a constant $c > 0$ independent of $t \in [0, T]$ such that*

$$\|\partial^\bullet \eta - \partial_h^\bullet \eta\|_{L^2(\Omega, L^2(\Gamma_0(t)))} \leq ch^2 \|\eta\|_{L^2(\Omega, H^1(\Gamma_0(t)))} \qquad \forall \eta \in L^2(\Omega, H^1(\Gamma_0(t))) \quad (3.6.32)$$

$$\|\nabla_{\Gamma_0(t)}(\partial^\bullet \eta - \partial_h^\bullet \eta)\|_{L^2(\Omega, L^2(\Gamma_0(t)))} \leq ch \|\eta\|_{L^2(\Omega, H^2(\Gamma_0(t)))} \qquad \forall \eta \in L^2(\Omega, H^2(\Gamma_0(t))). \quad (3.6.33)$$

Furthermore, such that for all $\Phi_h, \Psi_h \in L^2(\Omega, S_h^l(t))$ we have

$$\begin{aligned}|\tilde{a}(t, v; \varphi_h, \psi_h) - \tilde{a}(t, v_h; \varphi_h, \psi_h)| &\leq ch \|\varphi_h\|_{L^2(\Omega, H^1(\Gamma_0(t)))} \|\psi_h\|_{L^2(\Omega, H^1(\Gamma_0(t))))} \\ |\tilde{b}(t, v; \varphi_h, \psi_h) - \tilde{b}(t, v_h; \varphi_h, \psi_h)| &\leq ch \|\varphi_h\|_{L^2(\Omega, L^2(\Gamma_0(t)))} \|\psi_h\|_{L^2(\Omega, H^1(\Gamma_0(t))))}.\end{aligned}$$

Proof. The first two estimates immediately follow by observing that

$$\partial^\bullet \eta - \partial_h^\bullet \eta = \nabla_{\Gamma_0(t)} \cdot (v - v_h)$$

since $v - v_h$ is tangential to the surface. We may then apply the above estimate on the error between the discrete material velocity field and the smooth velocity, to obtain (3.6.32) and differentiate to obtain the second estimate (3.6.33). We next recall that

$$\begin{aligned}\tilde{a}(t, v; \varphi_h, \psi_h) &= \int_\Omega \int_{\Gamma_0(t)} \tilde{\mathcal{A}}(v) \nabla_{\Gamma_0(t)} \varphi_h \cdot \nabla_{\Gamma_0(t)} \psi_h \\ \tilde{b}(t, v; \varphi_h, \psi_h) &= \int_\Omega \int_{\Gamma_0(t)} \varphi_h \tilde{\mathcal{B}}(v) \cdot \nabla_{\Gamma_0(t)} \psi_h,\end{aligned}$$

with

$$\begin{aligned}\tilde{\mathcal{A}}(v) &= \partial^\bullet(\mathcal{A}) + \mathcal{A}(\nabla_{\Gamma_0(t)} \cdot v) - \mathcal{A} \nabla_{\Gamma_0(t)} v^\top - \nabla_{\Gamma_0(t)} \mathcal{A} \\ \tilde{\mathcal{B}}(v) &= \partial^\bullet(\mathcal{B}) + \mathcal{B}(\nabla_{\Gamma_0(t)} \cdot v) - \nabla_{\Gamma_0(t)} v \mathcal{B}.\end{aligned}$$

Hence subtracting the terms gives respectively

$$\tilde{\mathcal{A}}(v) - \tilde{\mathcal{A}}(v_h) = \partial^\bullet \mathcal{A} - \partial_h^\bullet \mathcal{A} + \mathcal{A} \nabla_{\Gamma_0(t)} \cdot (v - v_h) - \mathcal{A}(\nabla_{\Gamma_0(t)}(v - v_h))^\top - \nabla_{\Gamma_0(t)}(v - v_h)\mathcal{A}$$

and

$$\tilde{\mathcal{B}}(v) - \tilde{\mathcal{B}}(v_h) = \partial^\bullet \mathcal{B} - \partial_h^\bullet \mathcal{B} + \mathcal{B}(\nabla_{\Gamma_0(t)} \cdot (v - v_h)) - \nabla_{\Gamma_0(t)}(v - v_h)\mathcal{B}.$$

which may be bound with the uniform estimates of the random coefficients derived in Lemma 3.4.1, the estimates on the velocity field approximation given in Lemma 3.6.7 and Lemma 3.6.8, to give the stated bounds. □

We conclude this section by verifying that the remaining assumption (R) on regularity of an associated dual problem is indeed satisfied for our particular problem. The associated dual problem is as follows.

Problem 3.6.2 (The dual problem). *Given $\hat{f} \in L^2(\Omega, L^2(\Gamma_0(t)))$, find $\hat{u} \in L^2(\Omega, H^1(\Gamma_0(t)))$ such that*

$$\int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0(t)} \hat{u} \cdot \nabla_{\Gamma_0(t)} \hat{\varphi} + \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \hat{\varphi} \nabla_{\Gamma_0(t)} \hat{u} \cdot (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0})$$
$$+ \int_\Omega \int_{\Gamma_0(t)} \kappa \sqrt{g_{\Gamma_0}} \hat{u} \hat{\varphi} = \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \hat{f} \hat{\varphi} \qquad \forall \hat{\varphi} \in L^2(\Omega, H^1(\Gamma_0(t))).$$

We observe that this is precisely the mean-weak formulation of the following random elliptic equation on the reference surface $\Gamma_0(t)$

$$-\frac{1}{\sqrt{g_{\Gamma_0}}} \nabla_{\Gamma_0(t)} \cdot \left(\sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0(t)} \hat{u} \right) + \nabla_{\Gamma_0(t)} \hat{u} \cdot (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0}) + \kappa \hat{u} = \hat{f} \quad \text{on } \Gamma_0(t).$$

If we define the pathwise solution on the corresponding realisation of the random surface $\Gamma_\omega(t)$ by $u_\omega \circ \phi_\omega(t) = \hat{u}_\omega$, where we recall $\phi_\omega(t) : \Gamma_0(t) \to \Gamma_\omega(t)$ is the prescribed stochastic domain mapping, then with the chain rule

$$\nabla_{\Gamma_0(t)} \hat{u}_\omega = \nabla_{\Gamma_0(t)} \phi_\omega(t)^\top \nabla_{\Gamma_\omega(t)} u_\omega \circ \phi_\omega(t)$$

we may reformulate the associated dual problem onto the random surface $\Gamma_\omega(t)$, to obtain the following elliptic equation

$$-\Delta_{\Gamma_\omega(t)} u_\omega + \nabla_{\Gamma_\omega(t)} u_\omega \cdot b_\omega(t) + \kappa u_\omega = f_\omega \quad \text{on } \Gamma_\omega(t).$$

Here the random tangential advection $b_\omega(t) \circ \phi_\omega(t)^{-1} : \Gamma_\omega(t) \to T\Gamma_\omega(t)$ on the random surface is given by

$$b_\omega(t) := \nabla_{\Gamma_0(t)} \phi_\omega(t) (v_{\tau,arb}^{\Gamma_0} - v_{\tau,corr}^{\Gamma_0}) \quad \text{on } \Gamma_0(t),$$

and where have set $f_\omega \circ \phi_\omega(t) = \hat{f}_\omega$ to be the push-forward of the realisation of the data. As we have already derived uniform bounds on

$$\|b_\omega(t)\|_{L^\infty(\Gamma_0(t))} \leq C$$

for a constant $C > 0$ independent of ω and t by our assumptions of the stochastic domain mapping, and since realisations of the random surface $\Gamma_\omega(t)$ are assumed to be C^2, it then follows from standard elliptic theory that the realisations of the pathwise solution belong to space $u_\omega \in H^2(\Gamma_\omega(t))$. Furthermore, we may estimate the H^2 semi-norm by interchanging tangential derivatives

$$\begin{aligned}
|u_\omega|_{H^2(\Gamma_\omega(t))} &\leq \|\Delta_{\Gamma_\omega(t)} u_\omega\|_{L^2(\Gamma_\omega(t))} + c(\omega,t)|u_\omega|_{H^1(\Gamma_\omega(t)} \\
&\leq \|f_\omega - \kappa u_\omega - \nabla_{\Gamma_\omega(t)} u_\omega \cdot b_\omega(t)\|_{L^2(\Gamma_\omega(t)))} + c(\omega,t)|u_\omega|_{H^1(\Gamma_\omega(t))} \\
&\leq C\|f_\omega\|_{L^2(\Gamma_\omega(t))}
\end{aligned}$$

and applying the stability bound on $\|u_\omega\|_{H^1(\Gamma_\omega(t)}$. Note here that the random constant $c(\omega,t)$ is given by

$$c(\omega,t)^2 = \|H^{\Gamma_\omega(t)}\mathcal{H}^{\Gamma_\omega(t)} - 2\left(\mathcal{H}^{\Gamma_\omega(t)}\right)^2\|_{L^\infty(\Gamma_{\omega(t)})}$$

and may be shown to be uniformly bounded by a constant independent of ω and t by the our given assumptions on the stochastic domain mapping, as was discussed for the random elliptic surface equation in the previous chapter.

Theorem 3.6.1 (Convergence rate). *Let us assume that the initial condition $u_{h,0} = U^l_{h,0}$ for the semi-discrete problem satisfies*

$$\|u_0 - u_{h,0}\|_{L^2(\Omega,L^2(\Gamma_0(0)))} \leq ch^2.$$

Then we the following error estimate holds

$$\sup_{t\in[0,T]} \|E[u(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,L^2(\Gamma_0(t)))} \lesssim h^2 + \frac{1}{\sqrt{M}} \tag{3.6.34}$$

$$\int_0^T \|E[u(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M,H^1(\Gamma_0(t)))}\, dt \lesssim h + \frac{1}{\sqrt{M}}. \tag{3.6.35}$$

3.7 A discretisation of the reformulated coupled advection-diffusion system on the evolving bulk-surface reference domain

In this section, we continue onto our second model problem of a coupled advection-diffusion system on a randomly evolving bulk-surface. In particular, we will propose a semi-discretisation of the reformulated equations on the evolving reference bulk-surface, based upon the evolving finite element approach presented in [35]. Note that, the nodes of our mesh which approximates the evolving reference bulk-surface, will evolve by an arbitrary velocity field which was previously introduced and will be further discussed. We will now recap some of the key results and notation adopted for the reformulated problem as a point of reference for the subsequent analysis. Note that this application and the subsequent analysis which will follow, is motivated by and develops upon the results presented in [34, 35] which consider a specific deterministic case. We extend these results presented to our particular random geometric setting and prove in a similar fashion to [35] that the all the necessary assumptions of the abstract numerical analysis are indeed satisfied.

3.7.1 Summary of the reformulated system on the reference bulk-surface

The computational reference domain for the extended domain mapping method is comprised of an evolving compact surface $\Gamma_0(t) \subset \mathbb{R}^{n+1}$ which bounds an open interior bulk domain $D_0(t)$ and whose evolution is characterised by the given normal velocity field $w_\nu^{\Gamma_0}$. An arbitrary material flow was introduced on the bulk-surface domain denoted by

$$w = \begin{cases} w_{arb} & \text{in } D_0(t) \\ w_\nu^{\Gamma_0} + w_{\tau,arb}^{\Gamma_0} & \text{on } \Gamma_0(t), \end{cases}$$

and was assumed to describe a sufficiently smooth flow over the whole bulk-surface $\overline{D_0(t)}$, see Assumption 3.4.9 for further details. The mean-weak formulation was then of the following form for the given random functions spaces

$$H(t) = L^2(\Omega, L^2(D_0(t)) \times L^2(\Gamma_0(t))) \qquad V(t) = L^2(\Omega, H^1(D_0(t)) \times H^1(\Gamma_0(t))).$$

Problem 3.7.1 (Mean-weak formulation). *Given $(u_0, v_0) \in V_0$, find a pair $(u, v) \in W(V, H)$ such that for a.e. $t \in [0, T]$*

$$m(t; (\partial_w^\bullet u, \partial_w^\bullet v), (\varphi, \xi)) + g(t, w; (u, v), (\varphi, \xi)) + a(t; (u, v), (\varphi, \xi)) + b(t; (u, v), (\varphi, \xi)) = 0$$

for all $(\varphi, \xi) \in V(t)$, and which furthermore satisfies the initial condition

$$(u(0), v(0)) = (u_0, v_0) \quad \text{in } H_0.$$

Here the bilinear forms associated to the continuous problem are given by

$$\begin{aligned} m(t; (u, v), (\varphi, \xi)) &= \alpha \int_\Omega \int_{D_0(t)} \sqrt{g} u\varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} v\xi \\ a(t; (u, v), (\varphi, \xi)) &= \alpha \int_\Omega \int_{D_0(t)} \mathcal{A}\nabla u \cdot \nabla\varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \mathcal{A}_{\Gamma_0} \nabla_{\Gamma_0(t)} v \cdot \nabla_{\Gamma_0(t)} \xi \\ &\quad + \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} (\alpha u - \beta v)(\alpha\varphi - \beta\xi) \\ b(t; (u, v), (\varphi, \xi)) &= \alpha \int_\Omega \int_{D_0(t)} u\mathcal{B} \cdot \nabla\varphi + \beta \int_\Omega \int_{\Gamma_0(t)} v\mathcal{B}_{\Gamma_0} \cdot \nabla_{\Gamma_0(t)} \xi \end{aligned}$$

$$\begin{aligned} g(t, w; &(u, v), (\varphi, \xi)) \\ &= \alpha \int_\Omega \int_{D_0(t)} (\partial_w^\bullet(\sqrt{g}) + \sqrt{g}\nabla \cdot w)\, u\varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \left(\partial_w^\bullet(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}} \nabla_{\Gamma_0(t)} \cdot w\right) v\xi. \end{aligned}$$

$$\tilde{a}(t,w;(u,v),(\varphi,\xi)) = \alpha\int_\Omega\int_{D_0(t)} \tilde{\mathcal{A}}(w)\nabla u\cdot\nabla\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} \tilde{\mathcal{A}}_{\Gamma_0}(w)\nabla_{\Gamma_0(t)}v\cdot\nabla_{\Gamma_0(t)}\xi$$
$$+\int_\Omega\int_{\Gamma_0(t)}(\alpha u-\beta v)(\alpha\varphi-\beta\xi)\partial^\bullet_w(\sqrt{g_{\Gamma_0}}) \tag{3.7.1}$$

and

$$\tilde{b}(t,w;(u,v),(\varphi,\xi)) = \alpha\int_\Omega\int_{D_0(t)} u\tilde{\mathcal{B}}(w)\cdot\nabla\varphi + \beta\int_\Omega\int_{\Gamma_0(t)} v\tilde{\mathcal{B}}_{\Gamma_0}(w)\cdot\nabla_{\Gamma_0(t)}\xi, \tag{3.7.2}$$

where the random coefficients are given by

$$\begin{aligned}
\mathcal{A} &= \sqrt{g}G^{-1} & \mathcal{A}_{\Gamma_0} &= \sqrt{g_{\Gamma_0}}G^{-1}_{\Gamma_0}\\
\mathcal{B} &= \sqrt{g}(w-w_{corr}) & \mathcal{B}_{\Gamma_0} &= \sqrt{g_{\Gamma_0}}(w^{\Gamma_0}_{\tau,arb}-w^{\Gamma_0}_{\tau,corr})
\end{aligned}$$

and furthermore by

$$\begin{aligned}
\tilde{\mathcal{A}}(w) &= \partial^\bullet_w(\mathcal{A}) + \mathcal{A}(\nabla\cdot w) - \nabla w\mathcal{A} - \mathcal{A}\nabla w^\top\\
\tilde{\mathcal{A}}_{\Gamma_0}(w) &= \partial^\bullet_w(\mathcal{A}_{\Gamma_0}) + \mathcal{A}_{\Gamma_0}(\nabla_{\Gamma_0(t)}\cdot w) - \nabla_{\Gamma_0(t)}w\mathcal{A}_{\Gamma_0} - \mathcal{A}_{\Gamma_0}\nabla_{\Gamma_0(t)}w^\top\\
\tilde{\mathcal{B}}(w) &= \partial^\bullet_w\mathcal{B} + \mathcal{B}(\nabla\cdot w) - \nabla w\mathcal{B}\\
\tilde{\mathcal{B}}_{\Gamma_0}(w) &= \partial^\bullet_w\mathcal{B}_{\Gamma_0} + \mathcal{B}_{\Gamma_0}(\nabla_{\Gamma_0(t)}\cdot w) - \nabla_{\Gamma_0(t)}w\mathcal{B}_{\Gamma_0}.
\end{aligned}$$

For convenience, we will often express the strong material derivative with respect to the ALE material velocity field w, by $\partial^\bullet$ instead of $\partial^\bullet_w$, provided it is clear which velocity field is being referred to in the given application. Uniform estimates on the random bulk and surface coefficients as well as their material derivatives, are derived in Lemma 3.4.2 and Lemma 3.4.3. We will now continue by proposing our semi-discretisation of the reformulated system. We will begin by first decribing the computational discrete domain which will approximate the smooth evolving bulk-surface reference domain, which will be based upon the approach presented in [35].

3.7.2 Evolving discrete domain

We approximate the smooth evolving open bulk domain $D_0(t)\subset\mathbb{R}^{n+1}$ by a polyhedral domain

$$D_h(t) = \bigcup_{K(t)\in\hat{\mathcal{T}}_h(t)} K(t) \tag{3.7.3}$$

consisting of closed $(n+1)$-simplices whose maximum diameter is uniformly bounded by h and such that the triangulation $\hat{\mathcal{T}}_h(t)$ is quasi-uniform in time, i.e. in ball radius of each simplex $\sigma(K(t))$ is uniformly bounded below by $\sigma(K(t))\geq ch$. We assume that the vertices of the triangulation which lie on the discrete boundary $\partial D_h(t)$ also lie on the smooth boundary $\Gamma_0(t)$ such that $\partial D_h(t)$ interpolates the smooth boundary. We denote the induced discrete surface approximating $\Gamma_0(t)$ by

$$\Gamma_h(t) = \bigcup_{T(t)\in\mathcal{T}_h(t)} T(t) \qquad \partial\Gamma_h(t) = \emptyset. \tag{3.7.4}$$

and assume that all the assumptions given in the previous section hold for $\Gamma_h(t)$. In particular, we assume that $\Gamma_h(t)$ is contained within a neighbourhood of the smooth boundary $\Gamma_0(t)$ in which the Fermi coordinates are well-defined and furthermore that the restriction of the projection mapping $a(\cdot,t) : \Gamma_h(t) \to \Gamma_0(t)$ is a bijection, so we do not have double covering of curved surface simplices. We may then as previously seen, define lifts of functions between the discrete and continuous surface by the relation

$$f^l(a(x,t),t) = f(x,t) \qquad x \in \Gamma_h(t).$$

It is further possible to construct a diffeomorphic mapping

$$\tilde{\Lambda}_K(\cdot,t) : K(t) \to K^e(t)$$

between boundary simplices $K(t) \in \hat{\mathcal{T}}_h(t)$, i.e. simplices with at least two vertices on the discrete surface, and the exact curved simplices which we shall denote by $K^e(t)$, in such a way that extends the projection operator $a(\cdot,t) : \Gamma_h(t) \to \Gamma_0(t)$

$$\tilde{\Lambda}_K(x,t) = a(x,t) \qquad x \in \Gamma_h(t).$$

In other words, the diffeomorphic mapping coincides with the lifting mapping $a(\cdot,t)$ on the boundary of the discrete surface, and is extended into the interior to define the curved simplices $K^e(t) = \tilde{\Lambda}_K(K(t),t)$. Precise details of such a mapping may be found in [35]. A lifting mapping $\Lambda_h(\cdot,t) : D_h(t) \to D_0(t)$ may now be defined between the discrete and continuous bulk domains as follows

$$\Lambda_h(\cdot,t)|_K = \begin{cases} \tilde{\Lambda}_K(\cdot,t) & K(t) \text{ is a boundary simplex,} \\ id|_{K(t)} & K(t) \text{ is a interior simplex.} \end{cases} \tag{3.7.5}$$

We denote the corresponding the lift of function defined over the discrete bulk domain onto the smooth bulk by the given lifting mapping, by

$$f^l(\Lambda_h(x,t),t) = f(x,t) \qquad x \in D_h(t)$$

without any ambiguity to which lifting map is referred to, as both lifting mapping coincide on the discrete boundary $\Gamma_h(t)$. We finally assume that all of the vertices $\{X_j(t)\}_{j=1}^N$ of discrete domain $\overline{D_h(t)}$ evolve by the prescribe ALE velocity field w,

$$X_j'(t) = w(X_j(t),t) \quad \forall t \in [0,T].$$

3.7.3 Evolving finite element spaces and the discrete material derivatives

We introduce piecewise linear finite element spaces on the evolving discrete surface and bulk domain as follows

$$
\begin{aligned}
V_h(t) &= \{\Phi_h \in C^0(D_h(t)) \,|\, \Phi_h|_K \in P^1(K) \quad \forall K \in \hat{\mathcal{T}}_h(t)\} \\
S_h(t) &= \{\chi_h \in C^0(\Gamma_h(t)) \,|\, \chi_h|_T \in P^1(T) \quad \forall T \in \mathcal{T}_h(t)\}
\end{aligned}
$$

and denote their corresponding lifted spaces by

$$
\begin{aligned}
V_h^l(t) &= \{\varphi_h = \Phi_h^l \,|\, \Phi_h \in V_h(t)\} \\
S_h^l(t) &= \{\xi_h = \chi_h^l \,|\, \chi_h \in S_h(t)\}.
\end{aligned}
$$

We observe that $V_h^l(t) \subset H^1(D_0(t))$ and that $S_h^l(t) \subset H^1(\Gamma_0(t))$. We next introduce a discrete material velocity field over the whole discrete bulk-surface as the Lagrangian interpolant of the ALE velocity w,

$$
W_h(x,t) = I_h w(x,t) \qquad x \in \overline{D_h(t)}. \tag{3.7.6}
$$

We define the associated discrete material derivatives on the discrete bulk and discrete surface element-wise via the equations

$$
\begin{aligned}
\partial_h^\bullet \Phi_h|_{K(t)} &= (\Phi_{h,t} + \nabla \Phi_h \cdot W_h)\,|_{K(t)} \\
\partial_h^\bullet \chi_h|_{T(t)} &= \big(\chi_{h,t} + \nabla_{\Gamma_h(t)} \chi_h \cdot W_h\big)\,|_{T(t)}.
\end{aligned}
$$

Under the lifting mapping $\Lambda_h(\cdot,t) : \overline{D_h(t)} \to \overline{D_0(t)}$, there is an induced material flow on the continuous bulk-surface $\overline{D_0(t)}$ which we shall denote by w_h. The discrete material derivatives corresponding to this velocity field will be denote by

$$
\begin{aligned}
\partial_h^\bullet \varphi|_{K^l(t)} &= (\varphi_{h,t} + \nabla \varphi_h \cdot w_h)\,|_{K^l(t)} \\
\partial_h^\bullet \xi|_{T^l(t)} &= \big(\xi_{h,t} + \nabla_{\Gamma_0(t)} \xi \cdot w_h\big)\,|_{T^l(t)}.
\end{aligned}
$$

3.7.4 The semi-discrete problem

In the abstract framework, the evolving finite element space is given by

$$
\mathcal{V}_h(t) = V_h(t) \times S_h(t)
$$

and the two norms which equip the space are given by

$$
\begin{aligned}
\|(\Phi_h, \chi_h)\|_{\mathcal{H}_h(t)} &= \|(\Phi_h, \chi_h)\|_{L^2(D_h(t)) \times L^2(\Gamma_h(t))} \\
\|(\Phi_h, \chi_h)\|_{\mathcal{V}_h(t)} &= \|(\Phi_h, \chi_h)\|_{H^1(D_h(t)) \times H^1(\Gamma_h(t))}.
\end{aligned}
$$

We observe that the above norms clearly satisfy the following assumption given in the abstract setting.

$$\|(\Phi_h, \chi_h)\|_{\mathcal{H}_h(t)} \leq \|(\Phi_h, \chi_h)\|_{\mathcal{V}_h(t)}.$$

Furthermore, for our problem, the random discrete function spaces are given by

$$\begin{aligned} H_h(t) &= L^2(\Omega, V_h(t) \times S_h(t)) \\ V_h(t) &= L^2(\Omega, V_h(t) \times S_h(t)) \end{aligned}$$

and the abstract push-forward operator

$$\phi_t^h : \mathcal{V}_h(0) \to \mathcal{V}_h(t)$$

describing the evolution of the finite element space is given by the flow map associated with the ALE velocity field w, i.e. for $(u_0, v_0) \in V_h(0) \times S_h(0)$

$$\phi_t^h(u_0, v_0)(G_h(x,t), G_h(y,t)) = (u_0(x), v_0(y)) \quad \text{for } x \in D_h(t), y \in \Gamma_h(t))$$

where the flow map $G_h(\cdot, t) : \overline{D_h(0)} \to \overline{D_h(t)}$ of the discrete bulk-surface is defined by

$$G_h'(x,t) = W_h(G_h(x,t), t) \qquad \forall x \in \overline{D_h(t)}.$$

Hence the abstract strong material derivative on the discrete bulk and surface is precisely given by the strong material derivatives with respect to the discrete velocity W_h, and will both be denoted by $\partial_h^\bullet$ without ambiguity. It follows the assumed smoothnes of the ALE velocity, see Assumption 3.4.9, that both of the pairs (H_h, ϕ_t^h) and (V_h, ϕ_t^h) are compatible. We next introduce the discrete analogues of the bilinear forms associated to the continuous problem, by defining for $(U_h, V_h), (\Phi_h, \chi_h) \in H_h(t), V_h(t)$

$$\begin{aligned} m_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= \alpha \int_\Omega \int_{D_h(t)} \sqrt{g^{-l}} U_h \Phi_h + \beta \int_\Omega \int_{\Gamma_h(t)} \sqrt{g_{\Gamma_0}^{-l}} V_h \chi_h \\ a_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= \alpha \int_\Omega \int_{D_h(t)} \mathcal{A}^{-l} \nabla U_h \cdot \nabla \Phi_h + \beta \int_\Omega \int_{\Gamma_h(t)} \mathcal{A}_{\Gamma_0}^{-l} \nabla_{\Gamma_h(t)} V_h \cdot \nabla_{\Gamma_h(t)} \chi_h \\ &\quad + \int_\Omega \int_{\Gamma_h(t)} \sqrt{g_{\Gamma_0}^{-l}} (\alpha U_h - \beta V_h)(\alpha \Phi_h - \beta \chi_h) \\ b_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= \alpha \int_\Omega \int_{D_h(t)} U_h \mathcal{B}^{-l} \cdot \nabla \Phi_h + \beta \int_\Omega \int_{\Gamma_h(t)} V_h \mathcal{B}_{\Gamma_0}^{-l} \cdot \nabla_{\Gamma_h(t)} \chi_h \\ g_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) & \\ &= \alpha \int_\Omega \int_{D_h(t)} \left(\partial_h^\bullet \left(\sqrt{g^{-l}} \right) + \sqrt{g^{-l}} \nabla \cdot W_h \right) U_h \Phi_h \\ &\quad + \beta \int_\Omega \int_{\Gamma_h(t)} \left(\partial_h^\bullet \left(\sqrt{g_{\Gamma_0}^{-l}} \right) + \sqrt{g_{\Gamma_0}^{-l}} \nabla_{\Gamma_0(t)} \cdot W_h. \right) V_h \chi_h. \end{aligned}$$

The semi-discrete problem for the reformulated coupled system is now as follows.

Problem 3.7.2 (Semi-discrete problem)**.** *Given* $(U_{h,0}, V_{h,0}) \in H_h(0)$*, find* $(U_h, V_h) \in \tilde{C}^1_{V_h}$ *such that for a.e.* $t \in [0,T]$,

$$\begin{aligned} m_h(t; (\partial^\bullet_h U_h, \partial^\bullet_h V_h), (\Phi_h, \chi_h)) + g_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) \\ + a_h(t; (U_h, V_h), (\Phi_h, \chi_h)) + b_h(t; (U_h, V_h), (\Phi_h, \chi_h)) = 0 \end{aligned}$$

for all $(\Phi_h, \chi_h) \in V_h(t)$ *and which furthermore satisfies the initial condition*

$$(U_h(0), V_h(0)) = (U_{h,0}, V_{h,0}) \quad \textit{in } H_h(0).$$

In order to verify that the semi-discrete problem is well-posed, we first check all the assumptions given abstract analysis are satisfied. We first note that by the derived uniform bounds on the random bulk and surface coefficients given in Lemma 3.4.2 and Lemma 3.4.3, we obtain the estimates $(M_h 1), (A_h 2), (A_h 2)$ and $(B_h 1)$. We next note by the discrete Leibniz transport formula Lemma 3.6.1 applied element-wise, that we have the following discrete transport properties

$$\begin{aligned} \frac{d}{dt} m_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= m_h(t; (\partial^\bullet_h U_h, \partial^\bullet_h V_h), (\Phi_h, \chi_h)) \\ &\quad + m_h(t; (U_h, V_h), (\partial^\bullet_h \Phi_h, \partial^\bullet_h \chi_h)) + g_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) \end{aligned} \tag{3.7.7}$$

$$\begin{aligned} \frac{d}{dt} a_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= a_h(t; (\partial^\bullet_h U_h, \partial^\bullet_h V_h), (\Phi_h, \chi_h)) \\ &\quad + a_h(t; (U_h, V_h), (\partial^\bullet_h \Phi_h, \partial^\bullet_h \chi_h)) + \tilde{a}_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) \end{aligned} \tag{3.7.8}$$

$$\begin{aligned} \frac{d}{dt} b_h(t; (U_h, V_h), (\Phi_h, \chi_h)) &= b_h(t; (\partial^\bullet_h U_h, \partial^\bullet_h V_h), (\Phi_h, \chi_h)) \\ &\quad + b_h(t; (U_h, V_h), (\partial^\bullet_h \Phi_h, \partial^\bullet_h \chi_h)) + \tilde{b}_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) \end{aligned} \tag{3.7.9}$$

where the arising bilinear forms are given by

$$\begin{aligned} \tilde{a}_h(t, W_h; &(U_h, V_h), (\Phi_h, \chi_h)) \\ &= \alpha \int_\Omega \int_{D_h(t)} \tilde{\mathcal{A}}_h(W_h) \nabla U_h \cdot \nabla \Phi_h + \beta \int_\Omega \int_{\Gamma_h(t)} \tilde{\mathcal{A}}_{\Gamma_0,h}(W_h) \nabla_{\Gamma_h(t)} V_h \cdot \nabla_{\Gamma_h(t)} \chi_h \\ &+ \int_\Omega \int_{\Gamma_h(t)} \partial^\bullet_h \left(\sqrt{g^{-l}_{\Gamma_0}} \right) (\alpha U_h - \beta V_h)(\alpha \Phi_h - \beta \chi_h) \end{aligned}$$

and

$$\tilde{b}_h(t, W_h; (U_h, V_h), (\Phi_h, \chi_h)) = \alpha \int_\Omega \int_{D_h(t)} U_h \tilde{\mathcal{B}}_h(W_h) \cdot \nabla \Phi_h + \beta \int_\Omega \int_{\Gamma_h(t)} V_h \tilde{\mathcal{B}}_{\Gamma_0,h}(W_h) \cdot \nabla_{\Gamma_h(t)} \chi_h.$$

where the random coefficients are given by

$$\tilde{\mathcal{A}}_h(W_h) = \partial_h^\bullet(\mathcal{A}^{-l}) + \mathcal{A}^{-l}\nabla\cdot W_h - \mathcal{A}^{-l}\nabla W_h^\top - \nabla W_h\mathcal{A}^{-l}$$
$$\tilde{\mathcal{A}}_{\Gamma_0,h}(W_h) = \partial_h^\bullet(\mathcal{A}_{\Gamma_0}^{-l}) + \mathcal{A}_{\Gamma_0}^{-l}\nabla_{\Gamma_h(t)}\cdot W_h - \mathcal{A}_{\Gamma_0}^{-l}\nabla_{\Gamma_h(t)} W_h^\top - \nabla_{\Gamma_h(t)} W_h\mathcal{A}_{\Gamma_0}^{-l}$$

and

$$\tilde{\mathcal{B}}_h(W_h) = \partial_h^\bullet\left(\mathcal{B}^{-l}\right) + \mathcal{B}^{-l}\nabla\cdot W_h - \nabla W_h\mathcal{B}^{-l}$$
$$\tilde{\mathcal{B}}_{\Gamma_0,h}(W_h) = \partial_h^\bullet\left(\mathcal{B}_{\Gamma_0}^{-l}\right) + \mathcal{B}_{\Gamma_0}^{-l}\nabla_{\Gamma_h(t)}\cdot W_h - \nabla_{\Gamma_h(t)} W_h\mathcal{B}_{\Gamma_0}^{-l}.$$

Hence by the uniform bounds derived in Lemma 3.4.2 and Lemma 3.4.2, on the random bulk and surface coefficients and their respective material derivatives, we deduce that the following abstract assumptions $(M_h2), (A_h3)$ and (B_h2) all hold and therefore deduce that the proposed semi-discrete problem approximating the continuous mean-weak formulation is well-posed by Theorem 3.5.1. We next observe the following discrete transport properties on the smooth bulk-surface with respect to the discrete material velocity field w_h. We have

$$\begin{aligned}\frac{d}{dt}m(t;(u_h,v_h),(\varphi_h,\xi_h)) &= m(t;(\partial_h^\bullet u_h,\partial_h^\bullet v_h),(\varphi_h,\xi_h))\\ &\quad + m(t;(u_h,v_h),(\partial_h^\bullet\varphi_h,\partial_h^\bullet\xi_h)) + g(t,w_h;(u_h,v_h),(\varphi_h,\xi_h)) \qquad (3.7.10)\\ \frac{d}{dt}a(t;(u_h,v_h),(\varphi_h,\xi_h)) &= a(t;(\partial_h^\bullet u_h,\partial_h^\bullet v_h),(\varphi_h,\xi_h))\\ &\quad + a(t;(u_h,v_h),(\partial_h^\bullet\varphi_h,\partial_h^\bullet\xi_h)) + \tilde{a}(t,w_h;(u_h,v_h),(\varphi_h,\xi_h)) \qquad (3.7.11)\\ \frac{d}{dt}b(t;(u_h,v_h),(\varphi_h,\xi_h)) &= b(t;(\partial_h^\bullet u_h,\partial_h^\bullet v_h),(\varphi_h,\xi_h))\\ &\quad + b(t;(u_h,v_h),(\partial_h^\bullet\varphi_h,\partial_h^\bullet\xi_h)) + \tilde{b}(t,w_h;(u_h,v_h),(\varphi_h,\xi_h)) \qquad (3.7.12)\end{aligned}$$

where the bilinear forms $\tilde{a}(t,w_h;\cdot,\cdot)$ and $\tilde{b}(t,w_h;\cdot,\cdot)$ are defined as in (3.7.1) and (3.7.2) , for the random coefficients

$$\tilde{\mathcal{A}}(w_h) = \partial_h^\bullet(\mathcal{A}) + \mathcal{A}(\nabla\cdot w_h) - \nabla w_h\mathcal{A} - \mathcal{A}\nabla w_h^\top$$
$$\tilde{\mathcal{A}}_{\Gamma_0}(w_h) = \partial_h^\bullet(\mathcal{A}_{\Gamma_0}) + \mathcal{A}_{\Gamma_0}(\nabla_{\Gamma_0(t)}\cdot w_h) - \nabla_{\Gamma_0(t)} w_h\mathcal{A}_{\Gamma_0} - \mathcal{A}_{\Gamma_0}\nabla_{\Gamma_0(t)} w_h^\top$$

and

$$\tilde{\mathcal{B}}(w_h) = \partial_h^\bullet\mathcal{B} + \mathcal{B}(\nabla\cdot w_h) - \nabla w_h\mathcal{B}$$
$$\tilde{\mathcal{B}}_{\Gamma_0}(w_h) = \partial_h^\bullet\mathcal{B}_{\Gamma_0} + \mathcal{B}_{\Gamma_0}(\nabla_{\Gamma_0(t)}\cdot w_h) - \nabla_{\Gamma_0(t)} w_h\mathcal{B}_{\Gamma_0}.$$

3.7.5 Error analysis and a convergence result

We will now continue by deriving an error bound for our proposed numerical scheme approximating the continuous mean-weak formulation, by verifying that all the assumptions given in the abstract numerical analysis hold. This will follow a similar fashion to the application presented in [35] for a particular deterministic case, but where we extend and generalise the results of [30, 34, 35] to our given random setting. For our particular problem in mind, the smooth abstract function space will be given by

$$\mathcal{Z}(t) = H^2(D_0(t)) \times H^2(\Gamma_0(t))$$

and the spaces in which an abstract lifting map

$$\Lambda_h(\cdot, t) : \mathcal{Z}_{0,h}(t) \to \mathcal{Z}_0(t)$$

exists will be given by

$$\mathcal{Z}_{0,h}(t) = C^0(D_h(t)) \times C^0(\Gamma_h(t)) \quad \mathcal{Z}_0(t) = C^0(D_0(t)) \times C^0(\Gamma_0(t)),$$

with the lifting mapping defined component-wise by

$$\Lambda_h((\Phi_h, \chi_h), t) = (\Phi_h^l, \chi_h^l) \qquad \Phi_h \in V_h(t), \chi_h \in S_h(t).$$

It then follows by the derived estimates on the surface lifting mapping Lemma 3.6.2, in conjuction with the following estimates on the bulk lifting mapping which may be found in [35, Proposition 4.9], that the given lifting mapping for our evolving finite element spaces satisfies assumptions $(L1)$ and $(L2)$ given in the abstract analysis.

Lemma 3.7.1 (Bulk lift estimates). *Given any $\Phi_h \in V_h(t)$ with corresponding lift $\varphi_h \in V_h^l(t)$, we have the following estimates for constants $C_i > 0$ independent of $t \in [0, T]$,*

$$\begin{aligned} C_1 \|\Phi_h\|_{L^2(D_h(t))} &\leq \|\varphi_h\|_{L^2(D_0(t))} \leq C_2 \|\Phi_h\|_{L^2(D_h(t))} \\ C_3 \|\Phi_h\|_{H^1(D_h(t))} &\leq \|\varphi_h\|_{H^1(D_0(t))} \leq C_4 \|\Phi_h\|_{H^1(D_h(t))}. \end{aligned}$$

An interpolation operator mapping the smooth functions spaces

$$I_h : H^2(D_0(t)) \times H^2(\Gamma_0(t)) \to V_h^l(t) \times S_h^l(t)$$

into the respective lifted finite element spaces is defined component-wise as

$$I_h(\varphi, \xi) = ((\tilde{I}_h \varphi^{-l})^l, (\tilde{I}_h \xi^{-l})^l) \quad \varphi \in H^2(D_0(t)) \ \ \xi \in H^2(D_0(t))$$

with $\tilde{I}_h$ denoting the standard Lagrangian interpolant on the discrete domain. Combining our previous estimates on the surface interpolation operator, with the following bulk interpolation estimates given in [35, Lemm 7.11], we deduce that assumption (I) given in the abstract analysis is satisfied.

Lemma 3.7.2 (Bulk interpolation estimate). *Given any $\varphi \in H^2(D_0(t))$, there exists a constant*

$C > 0$ *independent of* $t \in [0, T]$ *for which the following interpolation estimate holds*

$$\|\varphi - I_h\varphi\|_{L^2(D_0(t))} + h\|\varphi - I_h\varphi\|_{H^1(D_0(t))} \leq Ch^2|\varphi|_{H^2(D_0(t))}.$$

We next continue onto the assumed geometric pertubation estimates given in the abstract analysis. As previously mentioned, our bounds on the geometric pertubations will be based upon the results presented in [34, 35], where we generalise these results to our particular random setting. Since the geometric estimates for the error in the approximation of the surface terms in the bilinear forms have previously been derived in the last example, we will now focus solely on the bulk and coupling terms. We first introduce some new notation to explicitly treat the corresponding bulk terms in our variational formulation. Specifically, we define

$$\begin{aligned}
m^D(t; u, \varphi) &= \int_\Omega \int_{D_0(t)} \sqrt{g} u\varphi \\
a^D(t; u, \varphi) &= \int_\Omega \int_{D_0(t)} \mathcal{A}\nabla u \cdot \nabla\varphi \\
b^D(t; u, \varphi) &= \int_\Omega \int_{D_0(t)} u\mathcal{B} \cdot \nabla\varphi \\
g^D(t, w; u, \varphi) &= \int_\Omega \int_{D_0(t)} \left(\partial^\bullet(\sqrt{g}) + \sqrt{g}\nabla \cdot w\right) u\varphi \\
\tilde{a}^D(t, w; u, \varphi) &= \int_\Omega \int_{D_0(t)} \tilde{A}(w)\nabla u \cdot \nabla\varphi \\
\tilde{b}^D(t, w; u, \varphi) &= \int_\Omega \int_{D_0(t)} u\tilde{\mathcal{B}}(w) \cdot \nabla\varphi.
\end{aligned}$$

and denote the discrete analogues of the above bilinear forms defined over the discrete bulk by e.g. $m_h^D(t; U_h, \Phi_h)$. To obtains estimates on the order of the approximation of the discrete approximation of the continuous bulk bilinear forms, we require the following geometric estimates derived in [35, Proposition 4.7]

Lemma 3.7.3 (Geometric bulk estimates)**.** *Let* $\delta_h^{D_0} = |det(\nabla\Lambda_h)|$ *denote the volume element corresponding to the transformation from the discrete bulk domain to the continuous under the given lifting mapping* $\Lambda_h : D_h \to D_0$ *and set*

$$R_h^{D_0,1}(\omega) = \frac{1}{\delta_h^{D_0}} \left(\mathcal{A}^{-l}(\omega)\right)^{-1} \nabla\Lambda_h \mathcal{A}^{-l}(\omega)\nabla\Lambda_h^\top.$$

Then we have the following estimates for a constant $C > 0$ *independent of* $\omega \in \Omega$ *and* $t \in [0, T]$

$$\|\nabla G_h - I\|_{L^\infty(D_h(t))} \leq Ch \tag{3.7.13}$$

$$\|\delta_h^{D_0} - 1\|_{L^\infty(D_h(t))} \leq Ch \tag{3.7.14}$$

$$\|R_h^{D_0,1}(\omega) - I\|_{L^\infty(D_h(t))} \leq Ch. \tag{3.7.15}$$

We further require the following narrow band trace inequality to obtain higher order estimates as will later be seen.

Lemma 3.7.4 (Narrow band trace inequality). *Given $\delta > 0$ sufficiently small, let U_δ denote the narrow parallel band in the interior domain $D_0(t)$ around the boundary $\Gamma_0(t)$ defined by*

$$U_\delta(t) = \{x \in D_0(t) \,|\, -\delta < d^{\Gamma_0}(x,t) < 0\}. \tag{3.7.16}$$

Then for any $f \in H^1(D_0)$, we have the estimate, for a constant $C > 0$ independent of $t \in [0,T]$

$$\|f\|_{L^2(U_\delta(t))} \leq C\delta^{\frac{1}{2}}\|f\|_{H^1(D_0(t))}.$$

Combining these estimates, as originally proposed in [34], will lead to the following bounds on the geometric perturbations of the bulk bilinear forms.

Lemma 3.7.5 (Bulk geometric pertubations). *Given any $U_h, \Phi_h \in L^2(\Omega, V_h(t))$ with corresponding lifts $u_h = U_h^l, \varphi_h = \Phi_h^l$, we have the following estimate for constants $C > 0$ independent of $t \in [0,T]$*

$$\begin{aligned}
|m^D(t; u_h, \varphi_h) - m_h^D(t; U_h, \Phi_h)| &\leq Ch^2\|u_h\|_{L^2(\Omega,H_h(t))}\|\varphi_h\|_{L^2(\Omega,H_h(t))} \\
|a^D(t; u_h, \varphi_h) - a_h^D(t; U_h, \Phi_h)| &\leq Ch\|u_h\|_{L^2(\Omega,V_h(t))}\|\varphi_h\|_{L^2(\Omega,V_h(t))} \\
|b^D(t; u_h, \varphi_h) - b_h^D(t; U_h, \Phi_h)| &\leq Ch\|u_h\|_{L^2(\Omega,H_h(t))}\|\varphi_h\|_{L^2(\Omega,V_h(t))}.
\end{aligned}$$

Furthermore for $u, \varphi \in L^2(\Omega, H^2(D_0(t)))$ with inverse lifts $U = u^{-l}, \Phi = \varphi^{-l}$, we have the estimates

$$\begin{aligned}
|a^D(t; u, \varphi) - a_h^D(t; U, \Phi)| &\leq Ch^2\|u\|_{L^2(\Omega,H^2(D_0(t)))}\|\varphi\|_{L^2(\Omega,H^2(D_0(t)))} \\
|b^D(t; u, \varphi) - b_h^D(t; U, \Phi)| &\leq Ch^2\|u\|_{L^2(\Omega,H^2(D_0(t)))}\|\varphi\|_{L^2(\Omega,H^2(D_0(t)))}.
\end{aligned}$$

Proof. We first recall that lift of a function is defined by $\Phi_h^l(\Lambda_h(\cdot,t),t) = \Phi_h(\cdot,t)$. Therefore by the chain rule we have

$$\nabla\Lambda_h^\top \nabla\Phi_h^l = \nabla\Phi_h$$

and we may lift the discrete bilinear forms onto the smooth bulk domain to obtain the following

$$\begin{aligned}
m_h^D(t; U_h, \Phi_h) &= \int_\Omega \int_{D_0(t)} \sqrt{g}u_h\varphi_h \frac{1}{\delta_h^{D,l}} \\
a_h^D(t; U_h, \Phi_h) &= \int_\Omega \int_{D_0(t)} \mathcal{A}R_h^{D,1}\nabla u_h \cdot \nabla\varphi_h \\
b_h^D(t; U_h, \Phi_h) &= \int_\Omega \int_{D_0(t)} u_h R_h^{D,2}\mathcal{B} \cdot \nabla\varphi_h,
\end{aligned}$$

with

$$\begin{aligned} R_h^{D,1} &= \frac{1}{\delta_h^D}\mathcal{A}^{-1}\nabla\Lambda_h\mathcal{A}\nabla\Lambda_h^\top \\ R_h^{D,2} &= \frac{1}{\delta_h^D}\nabla\Lambda_h. \end{aligned}$$

Next, we observe by construction $\Lambda_h(x,t) = x$ for interior simplices. Thus if we define the set of all boundary simplices by

$$B_h(t) = \{T \in \mathcal{T}_h(t) \,|\, T \text{ is a boundary simplex }\}$$

and the corresponding lifted elements by

$$B_h^l(t) = \{T^l(t) \,|\, T(t) \in B_h(t)\},$$

then we have the following

$$\begin{aligned} m^D(t;u_h,\varphi_h) - m_h^D(t;U_h,\Phi_h) &= \int_\Omega \int_{B_h^l(t)} \sqrt{g}u_h\varphi_h\left(1-\frac{1}{\delta_h^D}\right) \\ a^D(t;u_h,\varphi_h) - a_h^D(t;U_h,\Phi_h) &= \int_\Omega \int_{B_h^l(t)} \mathcal{A}\left(I - R_h^{D,1}\right)\nabla u_h\cdot\nabla\varphi_h \\ b^D(t;u_h,\varphi_h) - b_h^D(t;U_h,\Phi_h) &= \int_\Omega \int_{B_h^l(t)} u_h\left(I - R_h^{D,2}\right)\mathcal{B}\cdot\nabla\varphi_h. \end{aligned}$$

Which consequently leads to the following estimates based on the previous geometric bulk estimates as well as the uniform bound on random coefficients

$$\begin{aligned} |m^D(t;u_h,\varphi_h) - m_h^D(t;U_h,\Phi_h)| &\le ch\|u_h\|_{L^2(\Omega,L^2(B_h^l(t)))}\|\varphi_h\|_{L^2(\Omega,L^2(B_h^l(t)))} \\ |a^D(t;u_h,\varphi_h) - a_h^D(t;U_h,\Phi_h)| &\le ch\|u_h\|_{L^2(\Omega,H^1(B_h^l(t)))}\|\varphi_h\|_{L^2(\Omega,H^1(B_h^l(t)))} \\ |m^D(t;u_h,\varphi_h) - m_h^D(t;U_h,\Phi_h)| &\le ch\|u_h\|_{L^2(\Omega,L^2(B_h^l(t)))}\|\varphi_h\|_{L^2(\Omega,H^1(B_h^l(t)))}. \end{aligned}$$

We next consider the parallel neighbourhood $U_\delta(t)$ of the smooth boundary $\Gamma_0(t)$ described in the narrow band trace inequality lemma, and choose the width as

$$0 < h < \delta \le ch.$$

We may subsequently employ the results of the aforementioned Lemma, to obtain an L^2-bound in this given neighbourhood for sufficiently smooth $f \in H^1(D_0(t))$ as follows

$$\|f\|_{L^2(U_\delta(t)} \le c\delta^{\frac{1}{2}}\|f\|_{H^1(D_0(t))} \le ch^{\frac{1}{2}}\|f\|_{H^1(D_0(t)}.$$

Applying this argument to the above estimates, making suitable assumptions of the test function regularity if needed, leads to the stated estimates. $\square$

We now proceed by establishing the geometric pertubation estimates for the bilinear forms which arise in the transport properties. For this we require estimates on the material derivative of geometric quantities, for which a proof may be found in [35, Lemma 7.13].

Lemma 3.7.6 (Geometric estimates)**.** *There exists constants $C > 0$ independent of $t \in [0, T]$ for which we have the following estimates,*

$$\sup_{t\in[0,t]} \|\partial_h^\bullet \nabla \Lambda_h\|_{L^\infty(D_h(t))} \leq Ch$$
$$\sup_{t\in[0,t]} \|\partial_h^\bullet \delta_h^D\|_{L^\infty(D_h(t))} \leq Ch.$$

The errors in the corresponding bilinear forms may now be bound as follows, where here we extend the arguments presented in [35] to our given random setting.

Lemma 3.7.7 (Bulk geometric pertubations for derivative terms)**.** *Given any $U_h, \Phi_h \in L^2(\Omega, V_h(t))$ with corresponding lifts $u_h = U_h^l, \varphi_h = \Phi_h^l$, we have the following estimate for constants $C > 0$ independent of $t \in [0, T]$*

$$|g^D(t, w_h; u_h, \varphi_h) - g_h^D(t, W_h; U_h, \Phi_h)| \leq Ch^2 \|u_h\|_{L^2(\Omega, H_h(t))} \|\varphi_h\|_{L^2(\Omega, H_h(t))}$$
$$|\tilde{a}^D(t, w_h; u_h, \varphi_h) - \tilde{a}_h^D(t, W_h; U_h, \Phi_h)| \leq Ch \|u_h\|_{L^2(\Omega, V_h(t))} \|\varphi_h\|_{L^2(\Omega, V_h(t))}$$
$$|\tilde{b}^D(t, w_h; u_h, \varphi_h) - \tilde{b}_h^D(t, W_h; U_h, \Phi_h)| \leq Ch \|u_h\|_{L^2(\Omega, H_h(t))} \|\varphi_h\|_{L^2(\Omega, V_h(t))}.$$

Furthermore for $u, \varphi \in L^2(\Omega, H^2(D_0(t)))$ with inverse lifts $U = u^{-l}, \Phi = \varphi^{-l}$, we have the estimates

$$|\tilde{a}^D(t, w_h; u, \varphi) - \tilde{a}_h^D(t, W_h; U, \Phi)| \leq Ch^2 \|u\|_{L^2(\Omega, H^2(D_0(t)))} \|\varphi\|_{L^2(\Omega, H^2(D_0(t)))}$$
$$|\tilde{b}^D(t, w_h; u, \varphi) - \tilde{b}_h^D(t, W_h; U, \Phi)| \leq Ch^2 \|u\|_{L^2(\Omega, H^2(D_0(t)))} \|\varphi\|_{L^2(\Omega, H^2(D_0(t)))}.$$

Proof. The stated estimate will follow a similar argument as was presented in the surface case. We first lift the discrete bilinear form onto the smooth bulk domain,

$$m_h^D(t; U_h, \Phi_h) = \int_\Omega \int_{D_0(t)} \sqrt{g} u_h \varphi_h \frac{1}{\delta_h^{D,l}},$$

and then differentiate in time applying the discrete transport property

$$\begin{aligned}\frac{d}{dt} m_h^D(t; U_h, \Phi_h) = &\int_\Omega \int_{D_0(t)} \sqrt{g} \partial_h^\bullet u_h \varphi_h \frac{1}{\delta_h^{D,l}} + \sqrt{g} u_h \partial_h^\bullet \varphi_h \frac{1}{\delta_h^{D,l}} \\ &+ \int_\Omega \int_{D_0(t)} \left(\partial_h^\bullet (\sqrt{g}) + \sqrt{g} \nabla \cdot w_h\right) \frac{1}{\delta_h^{D,l}} u_h \varphi_h \\ &+ \int_\Omega \int_{D_0(t)} \partial_h^\bullet \left(\frac{1}{\delta_h^{D,l}}\right) \sqrt{g} u_h \varphi_h.\end{aligned}$$

and finally pull back the first two integrals onto the discrete bulk to obtain

$$
\begin{aligned}
&= m_h^D(t; \partial_h^\bullet U_h, \Phi_h) + m_h^D(t; U_h, \partial_h^\bullet \Phi_h) + g^D(t, w_h; u_h, \varphi_h) \\
&- \int_\Omega \int_{D_0(t)} \left(1 - \frac{1}{\delta_h^{D,l}}\right) \left(\partial_h^\bullet(\sqrt{g}) + \sqrt{g}\nabla \cdot w_h\right) u_h \varphi_h \\
&+ \int_\Omega \int_{D_0(t)} \partial_h^\bullet \left(\frac{1}{\delta_h^{D,l}}\right) \sqrt{g} u_h \varphi_h.
\end{aligned}
$$

Comparing with the discrete transport property gives

$$
\begin{aligned}
g(t, w_h; u_h, \varphi_h) - g_h(t, W_h; U_h, \Phi_h) &= \int_\Omega \int_{B_h^l(t)} \left(1 - \frac{1}{\delta_h^{D,l}}\right) \left(\partial_h^\bullet(\sqrt{g}) + \sqrt{g}\nabla \cdot w_h\right) u_h \varphi_h \\
&- \int_\Omega \int_{B_h^l(t)} \partial_h^\bullet \left(\frac{1}{\delta_h^{D,l}}\right) \sqrt{g} u_h \varphi_h,
\end{aligned}
$$

which with the geometric estimates given in Lemma 3.7.7 combined with the derived uniform bounds on the random coefficients leads to the estimate

$$
|g(t, w_h; u_h, \varphi_h) - g_h(t, W_h; U_h, \Phi_h)| \leq c\|u_h\|_{L^2(\Omega, L^2(B_h^l(t)))} \|\varphi_h\|_{L^2(\Omega, L^2(B_h^l(t)))}.
$$

With the assumed regularity required for the narrow band trace inequality satisfied by our finite element functions, we obtain the desired bound of order h^2. For the bilinear form $a_h(t; \cdot, \cdot)$ and $b_h(t; \cdot, \cdot)$, the argument will follow closely to that was presented for the surface case. We therefore summarise the results as follows. We first begin with $a_h(t; \cdot, \cdot)$, where the lift is given by

$$
a_h(t; U_h, \Phi_h) = \int_\Omega \int_{D_0(t)} \mathcal{A} R_h^{D,1} \nabla u_h \cdot \nabla \varphi_h.
$$

Differentiating and comparing with the discrete tranpsort property leads to the following expression for the error

$$
\begin{aligned}
\tilde{a}(t, w_h; &u_h, \varphi_h) - \tilde{a}(t; W_h; U_h, \Phi_h) \\
&= \int_\Omega \int_{B_h^l(t)} \left(\partial_h^\bullet \mathcal{A} + (\nabla \cdot w_h)\mathcal{A} - \nabla w_h \mathcal{A}\right) \left(I - R_h^{D,1}\right) \nabla u_h \cdot \nabla \varphi_h \\
&- \int_\Omega \int_{B_h^l(t)} \mathcal{A} \left(R_h^{D,1} - I\right) \nabla w_h \nabla u_h \cdot \nabla \varphi_h \\
&- \int_\Omega \int_{B_h^l(t)} \mathcal{A} \partial_h^\bullet \left(R_h^{D,1}\right) \nabla u_h \cdot \nabla \varphi_h.
\end{aligned}
$$

Similarly for

$$
b_h(t; U_h, \Phi_h) = \int_\Omega \int_{D_0(t)} u_h \nabla \Lambda_h \mathcal{B} \cdot \nabla \varphi_h
$$

we obtain

$$
\begin{aligned}
\tilde{b}(t, w_h; u_h, &\varphi_h) - \tilde{b}_h(t, W_h; U_h, \Phi_h) \\
&= \int_\Omega \int_{D_0(t)} u_h \left(I - \nabla \Lambda_h\right) \left(\partial_h^\bullet \mathcal{B} - \mathcal{B} \nabla \cdot w_h\right) \cdot \nabla \varphi_h \\
&- \int_\Omega \int_{B_h^l(t)} u_h \nabla w_h \left(I - \nabla \Lambda_h\right) \mathcal{B} \cdot \nabla \varphi_h \\
&- \int_\Omega \int_{B_h^l(t)} u_h \partial_h^\bullet (\nabla \Lambda_h) \mathcal{B} \cdot \nabla \varphi_h.
\end{aligned}
$$

Hence in both cases, we may employ the uniform bounds on the random coefficients with the geometric estimates, to derive the stated bounds, where in particular, the higher order estimates will require use of the narrow band trace inequality as previously seen, and thus requires us to assume higher regularity on the test functions, as stated in this given lemma. □

We finally estimate the perturbation on the coupling term. For this, we introduce the following bilinear forms

$$
\begin{aligned}
a^{D,\Gamma}(t; (u, v), (\varphi, \xi)) &= \int_\Omega \int_{\Gamma_0(t)} (\alpha u - \beta v)(\alpha u - \beta v) \sqrt{g_{\Gamma_0}} \\
\tilde{a}^{D,\Gamma}(t, w_h; (u, v), (\varphi, \xi)) &= \int_\Omega \int_{\Gamma_0(t)} (\alpha u - \beta v)(\alpha \varphi - \beta v) \left(\partial_h^\bullet(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}} (\nabla_{\Gamma_0(t)} \cdot w_h)\right)
\end{aligned}
$$

where we have the discrete tranpsort property

$$
\begin{aligned}
\frac{d}{dt} a^{D,\Gamma}(t; (u, v), (\varphi, \xi)) = a^{D,\Gamma}&(t; (\partial_h^\bullet u, \partial_h^\bullet v), (\varphi, \xi))) + a^{D,\Gamma}(t; (u, v), (\partial_h^\bullet \varphi, \partial_h^\bullet \xi))) \\
&+ \tilde{a}^{D,\Gamma}(t, w_h; (u, v), (\varphi, \xi))).
\end{aligned}
$$

We denote their discrete analogues by $a_h^{D,\Gamma}(t; \cdot, \cdot)$ and $\tilde{a}_h^{D,\Gamma}(t, W_h; \cdot, \cdot)$, and bound the errors as follows.

Lemma 3.7.8 (Geometric estimates on coupling term)**.** *Given any $U_h, \Phi_h \in L^2(\Omega, V_h(t))$ with corresponding lifts $u_h = U_h^l, \varphi_h = \Phi_h^l$, we have the following estimate for constants $C > 0$ independent of $t \in [0, T]$*

$$
\begin{aligned}
|a^{D,\Gamma}(t; (u_h, v_h), &(\varphi_h, \xi_h)) - a_h^{D,\Gamma}(t; (U_h, V_h), (\Phi_h, \chi_h))| \\
&\leq Ch^2 \|(u_h, v_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))} \|(\varphi_h, \xi_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))}
\end{aligned}
$$

$$
\begin{aligned}
|\tilde{a}^{D,\Gamma}(t, w_h; (u_h, v_h), &(\varphi_h, \xi_h)) - \tilde{a}_h^{D,\Gamma}(t, W_h; (U_h, V_h), (\Phi_h, \chi_h))| \\
&\leq Ch^2 \|(u_h, v_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))} \|(\varphi_h, \xi_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))}.
\end{aligned}
$$

Proof. Lifting the discrete coupling bilinear form onto the smooth surface gives

$$a_h^{D,\Gamma}(t;(U_h,V_h),(\Phi_h,\chi_h)) = \int_\Omega \int_{\Gamma_0(t)} (\alpha u_h - \beta v_h)(\alpha\varphi_h - \beta\xi_h)\sqrt{g_{\Gamma_0}}\frac{1}{\delta_h^l}$$

and thus the error estimate

$$\begin{aligned}
&|a^{D,\Gamma}(t;(u_h,v_h),(\varphi_h,\xi_h)) - a_h^{D,\Gamma}(t;(U_h,V_h),(\Phi_h,\chi_h))| \\
&\quad\le C\|1-\frac{1}{\delta_h}\|_{L^\infty(\Gamma_h(t))}\|(u_h,v_h)\|_{L^2(\Omega,L^2(\Gamma_0(t))\times L^2(\Gamma_0(t)))}\|(\varphi_h,\xi_h)\|_{L^2(\Omega,L^2(\Gamma_0(t))\times L^2(\Gamma_0(t)))} \\
&\quad\le Ch^2\|(u_h,v_h)\|_{L^2(\Omega,H^1(D_0(t))\times L^2(\Gamma_0(t)))}\|(\varphi_h,\xi_h)\|_{L^2(\Omega,H^1(D_0(t))\times L^2(\Gamma_0(t)))},
\end{aligned}$$

where we have applied the geometric surface estimate, uniform bounds on random coefficients and also Poincare inequality on the bulk term. For the second estimate, we again summarise the key results as the method is similar to that of before. We differentiate the lift of the discrete bilinear form to obtain

$$\begin{aligned}
&\frac{d}{dt}a_h^{D,\Gamma}(t;(U_h,V_h),(\Phi_h,\xi_h)) \\
&= a^{D,\Gamma}(t;(\partial_h^\bullet u_h,\partial_h^\bullet v_h),(\varphi_h,\xi_h)) + a^{D,\Gamma}(t;(u_h,v_h),(\partial_h^\bullet\varphi_h,\partial_h^\bullet\xi_h)) + \tilde{a}^{D,\Gamma}(t,w_h;(u_h,v_h),(\varphi_h,\xi_h)) \\
&- \int_\Omega\int_{\Gamma_0(t)} (\alpha u_h - \beta v_h)(\alpha\varphi_h - \beta\xi_h)\left(\partial_h^\bullet(\sqrt{g_{\Gamma_0}}) + \sqrt{g_{\Gamma_0}}\nabla_{\Gamma_0(t)}\cdot w_h\right)\left(1-\frac{1}{\delta_h^l}\right) \\
&+ \int_\Omega\int_{\Gamma_0(t)} \partial_h^\bullet(\sqrt{g_{\Gamma_0}})(\alpha u_h - \beta v_h)(\alpha\varphi_h - \beta v_h).
\end{aligned}$$

Which leads to the following estimate

$$\begin{aligned}
&|\tilde{a}^{D,\Gamma}(t,w_h;(u_h,v_h),(\varphi_h,\xi_h)) - \tilde{a}_h^{D,\Gamma}(t,W_h;(U_h,V_h),(\Phi_h,\chi_h))| \\
&\qquad\qquad \le Ch^2\|(u_h,v_h)\|_{L^2(\Omega,L^2(\Gamma_0(t))\times L^2(\Gamma_0(t)))}\|(\varphi_h,\xi_h)\|_{L^2(\Omega,L^2(\Gamma_0(t))\times L^2(\Gamma_0(t)))},
\end{aligned}$$

and hence the stated result by applying Poincare inequality to the bulk term. □

We therefore may combine the estimates on the geometric pertubations of the smooth bilinear forms respectively relating to the surface terms, bulk terms and coupling terms, to deduce that all the stated assumptions $(G1)-(G9)$ in the abstract analysis hold. We next prove that the assumptions related to the approximation of the smooth velocity field w are all satisfied for our particular problem. We begin with a bound on the approximation of the smooth velocity field w by the discrete velocity w_h in the smooth bulk domain, for which a proof may be found in [35, Lemma 7.14].

Lemma 3.7.9 (Velocity estimates)**.** *There exists constants $C>0$ independent of $t\in[0,T]$ such that we have the following estimates*

$$\|w-w_h\|_{L^\infty(D_0(t))} + h\|\nabla(w-w_h)\|_{L^\infty(D_0(t))} \le Ch^2$$

and consequently

$$\|\partial^\bullet \varphi - \partial_h^\bullet \varphi\|_{L^2(\Omega, L^2(D_0(t)))} \leq ch^2 \qquad \forall \varphi(t) \in L^2(\Omega, H^1(D_0(t)))$$
$$\|\nabla(\partial^\bullet \varphi - \partial_h^\bullet \varphi)\|_{L^2(\Omega, L^2(D_0(t)))} \leq ch \qquad \forall \varphi(t) \in L^2(\Omega, H^2(D_0(t))).$$

The second and third estimates immediately follow by observing

$$\partial^\bullet \varphi - \partial_h^\bullet \varphi = \nabla \varphi \cdot (w - w_h)$$

and applying the estimate on velocity field approximation. This leads to the following bounds on the errors in the bilinears forms related to an approximation of the smooth velocity field.

Lemma 3.7.10 (Velocity estimates)**.** *Given any* $(u_h, v_h), (\varphi_h, \xi_h) \in L^2(\Omega, V_h^l(t) \times S_h^l(t))$*, we have the following bounds for constants* $C > 0$ *independent of* $t \in [0, T]$

$$\begin{aligned}|\tilde{a}(t, w;&(u_h, v_h), (\varphi_h, \xi_h)) - \tilde{a}(t, w_h; (u_h, v_h), (\varphi_h, \xi_h))| \\ &\leq Ch\|(u_h, v_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))}\|(\varphi_h, \xi_h)\|_{L^2(\Omega, H^1(D_0(t)) \times L^2(\Gamma_0(t)))}\end{aligned}$$

and

$$\begin{aligned}|\tilde{b}(t, w;&(u_h, v_h), (\varphi_h, \xi)h)) - \tilde{b}(t, w_h; (u_h, v_h), (\varphi_h, \xi_h))| \\ &\leq Ch\|(u_h, v_h)\|_{L^2(\Omega, L^2(D_0(t)) \times L^2(\Gamma_0(t)))}\|(\varphi_h, \xi_h)\|_{L^2(\Omega, H^1(D_0(t)) \times H^1(\Gamma_0(t)))}.\end{aligned}$$

Proof. We first recall that the bilinear form $\tilde{a}(t, w; \cdot, \cdot)$ is defined by

$$\begin{aligned}\tilde{a}(t, w;&(u_h, v_h), (\varphi_h, \xi_h)) \\ &= \alpha \int_\Omega \int_{D_0(t)} \tilde{\mathcal{A}}(w) \nabla u_h \cdot \nabla \varphi_h + \beta \int_\Omega \int_{\Gamma_0(t)} \tilde{\mathcal{A}}_{\Gamma_0}(w_h) \nabla_{\Gamma_0(t)} v_h \cdot \nabla_{\Gamma_0(t)} \xi_h \\ &= \int_\Omega \int_{\Gamma_0(t)} (\alpha u_h - \beta v_h)(\alpha \varphi_h - \beta \xi_h) \partial_w^\bullet \left(\sqrt{g_{\Gamma_0}}\right).\end{aligned}$$

Consequently, the error may be expressed as

$$\begin{aligned}&\tilde{a}(t, w; (u_h, v_h), (\varphi_h, \xi_h)) - \tilde{a}(t, w_h; (u_h, v_h), (\varphi_h, \xi_h)) \\ &= \alpha \int_\Omega \int_{D_0(t)} \left(\tilde{\mathcal{A}}(w) - \tilde{\mathcal{A}}(w_h)\right) \nabla u_h \cdot \nabla \varphi_h + \beta \int_\Omega \int_{\Gamma_0(t)} \left(\tilde{\mathcal{A}}_{\Gamma_0}(w) - \tilde{\mathcal{A}}_{\Gamma_0}(w_h)\right) \nabla_{\Gamma_0(t)} v_h \cdot \nabla_{\Gamma_0(t)} \xi_h \\ &= \int_\Omega \int_{\Gamma_0(t)} (\alpha u_h - \beta v_h)(\alpha \varphi_h - \beta \xi_h) \left(\partial_w^\bullet \left(\sqrt{g_{\Gamma_0}}\right) - \partial_{w_h}^\bullet \left(\sqrt{g_{\Gamma_0}}\right)\right).\end{aligned}$$

We next observe that

$$
\begin{aligned}
\tilde{\mathcal{A}}(w) - \tilde{\mathcal{A}}(w) &= \partial^\bullet_w \mathcal{A} - \partial^\bullet_{w_h} \mathcal{A} + \mathcal{A}\left(\nabla \cdot (w - w_h)\right) - \nabla(w - w_h)\mathcal{A} - \mathcal{A}\nabla(w - w_h) \\
\tilde{\mathcal{A}}_{\Gamma_0}(w) - \tilde{\mathcal{A}}_{\Gamma_0}(w) &= \partial^\bullet_w \mathcal{A}_{\Gamma_0} - \partial^\bullet_{w_h} \mathcal{A}_{\Gamma_0} + \mathcal{A}_{\Gamma_0}\left(\nabla_{\Gamma_0(t)} \cdot (w - w_h)\right) - \nabla_{\Gamma_0(t)}(w - w_h)\mathcal{A}_{\Gamma_0} \\
&\quad - \mathcal{A}_{\Gamma_0}\nabla_{\Gamma_0(t)}(w - w_h).
\end{aligned}
$$

Hence with the bounds previously derived on the approximation of the bulk velocity, combined with previous estimates on the approximation of the smooth velocity field on the surface $\Gamma_0(t)$ and uniform bounds on the random coefficients, we obtain our desired bounds. The second estimate follows a similar argument by recalling

$$
\tilde{b}(t, w; (u_h, v_h), (\varphi_h, \xi_h)) = \alpha \int_\Omega \int_{D_0(t)} u_h \tilde{\mathcal{B}}(w) \cdot \nabla \varphi_h + \beta \int_\Omega \int_{\Gamma_0(t)} v_h \tilde{\mathcal{B}}(w) \cdot \nabla_{\Gamma_0(t)} \xi_h,
$$

with

$$
\begin{aligned}
\tilde{\mathcal{B}}(w) &= \partial^\bullet_w \mathcal{B} + \mathcal{B}(\nabla \cdot w) - \nabla w \mathcal{B} \\
\tilde{\mathcal{B}}_{\Gamma_0}(w) &= \partial^\bullet_w \mathcal{B}_{\Gamma_0} + \mathcal{B}_{\Gamma_0}(\nabla_{\Gamma_0(t)} \cdot w) - \nabla_{\Gamma_0(t)} w \mathcal{B}_{\Gamma_0},
\end{aligned}
$$

and hence the difference may be estimated by the given estimates on the discrete approximation of the smooth velocity field. □

It therefore follows that the abstract assumptions $(V1)-(V4)$ all hold. We next continue by checking the assumed regularity (R) of the associated dual problem which reads as follows.

Problem 3.7.3 (Dual problem)**.** *Given $(f, f_{\Gamma_0}) \in L^2(\Omega, L^2(D_0(t)) \times L^2(\Gamma_0(t)))$, find $(u, v) \in L^2(\Omega, H^1(D_0(t)) \times H^1(\Gamma_0(t)))$ such that*

$$
\begin{aligned}
&\alpha \int_\Omega \int_{D_0(t)} \sqrt{g} G^{-1} \nabla u \cdot \nabla \varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0(t)} v \cdot \nabla_{\Gamma_0(t)} \xi \\
&+ \int_\Omega \int_{\Gamma_0(t)} (\alpha u - \beta v)(\alpha \varphi - \beta \xi) \sqrt{g_{\Gamma_0}} + \alpha \int_\Omega \int_{D_0(t)} \sqrt{g} \varphi \nabla u \cdot (w - w_{corr}) \\
&+ \beta \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} \xi \nabla_{\Gamma_0(t)} v \cdot (w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr}) + \kappa\alpha \int_\Omega \int_{D_0(t)} \sqrt{g} u \varphi + \kappa\beta \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} v \xi \\
&= \alpha \int_\Omega \int_{D_0(t)} \sqrt{g} f \varphi + \beta \int_\Omega \int_{\Gamma_0(t)} \sqrt{g_{\Gamma_0}} f_{\Gamma_0} \xi.
\end{aligned}
$$

for all $(\varphi, \xi) \in L^2(\Omega, H^1(D_0(t)) \times H^1(\Gamma_0(t)))$.

Motivated by [35], we begin by setting $\varphi = 0$ and observe that the weak surface solution solves the mean-formulation of the following random equation on $\Gamma_0(t)$

$$
-\frac{\beta}{\sqrt{g_{\Gamma_0}}} \nabla_{\Gamma_0(t)} \cdot \left(\sqrt{g_{\Gamma_0(t)}} G_{\Gamma_0}^{-1} \nabla_{\Gamma_0(t)} \cdot v \right) + \beta \nabla_{\Gamma_0(t)} v \cdot \left(w^{\Gamma_0}_{\tau,arb} - w^{\Gamma_0}_{\tau,corr} \right) + (\kappa\beta + \beta^2) v = \beta f_{\Gamma_0}.
$$

By a similar argument as was presented in the previous example, we deduce from the smoothness of the reference surface $\Gamma_0(t)$ and the uniform bounds on the random coefficients, that the weak solution v belongs to the space $v \in L^2(\Omega, H^2(\Gamma_0(t)))$ and satisfies the estimate

$$\|v\|_{L^2(\Omega,H^2(\Gamma_0(t)))} \le C\|f_{\Gamma_0}\|_{L^2(\Omega,L^2(\Gamma_0(t)))},$$

for a constant $C > 0$ independent of $t \in [0,T]$. For the bulk term, we choose $\xi = 0$ to deduce the bulk quantity is also a solution of a mean-weak formulation of the following random elliptic boundary value problem

$$-\frac{1}{\sqrt{g}}\nabla \cdot \left(\sqrt{g}G^{-1}\nabla u\right) + \nabla u \cdot (w - w_{corr}) + \kappa u = f \qquad \text{in } D_0(t)$$
$$\alpha u - \beta v + \frac{\sqrt{g}}{\sqrt{g_{\Gamma_0}}}G^{-1}\nu^{\Gamma_0} \cdot \nabla u = 0 \qquad \text{on } \Gamma_0(t).$$

Applying standard elliptic regularity results for $a.e.\omega$, and noting all the constants appearing are independent of ω by the derived uniform bounds on the random coefficients we deduce

$$\|u\|_{L^2(\Omega,H^2(D_0(t)))} \le C\left(\|f\|_{L^2(\Omega,L^2(D_0(t)))} + \|v\|_{L^2(\Omega,H^1(\Gamma_0(t)))}\right).$$

We may now combine the estimate to deduce $(u,v) \in L^2(\Omega, H^2(D_0(t)) \times H^2(\Gamma_0(t)))$ and furthermore satisfies the regularity estimate

$$\|(u,v)\|_{L^2(\Omega,H^2(D_0(t))\times H^2(\Gamma_0(t)))} \le C\|(f, f_{\Gamma_0})\|_{L^2(\Omega,L^2(D_0(t))\times L^2(\Gamma_0(t)))}.$$

Hence the abstract assumption (R) on the associated dual problem is satisfied. We have therefore shown all the listed assumption in abstract error analysis are in fact satisfied for our proposed semi-discretisation. We therefore have the following error bounds

$$\sup_{t\in[0,T]} \|(\mathbb{E}[u], \mathbb{E}[v]) - (E_M[u_h], E_M[v_h])\|_{L^2(\Omega^M,L^2(D_0(t))\times L^2(\Gamma_0(t)))} \lesssim h^2 + \frac{1}{\sqrt{M}}$$
$$\int_0^T \|(\mathbb{E}[u], \mathbb{E}[v]) - (E_M[u_h], E_M[v_h])\|_{L^2(\Omega^M,H^1(D_0(t))\times H^1(\Gamma_0(t)))} \lesssim h + \frac{1}{\sqrt{M}}.$$

3.8 Numerical results

In this section, we confirm by numerical experiments, the theoretical convergence rates of the two proposed schemes for the model problems under consideration. We note that all of the subsequent numerical results have been implemented in DUNE, see [1, 4] for further details.

3.8.1 Advection-diffusion on a randomly evolving surface

For the first advection-diffusion problem, we model the randomly evolving compact surface as a fluctuating graph over the unit sphere $\Gamma_0 = S^2$,

$$\{\Gamma_\omega(t) = \{x + h(\omega,t,x)\nu^{\Gamma_0}(x) \,|\, x \in \Gamma_0\}$$

where the random height function is prescribed by a truncated spherical harmonic series

$$h(\omega,t,x) = \epsilon_{tol}\sum_{l\leq 8}\sum_{|m|\leq l}\tilde{\lambda}_{l,m}(\omega,t)Y_l^m(\theta,\varphi) \qquad x = (sin\theta\, cos\varphi, sin\theta\, sin\varphi, cos\theta),$$

with $\epsilon_{tol} > 0$ denoting a parameter which controls the maximum deviation of the random surface. The random coefficients of the expansion above are each given by a truncated fourier series in time

$$\tilde{\lambda}_{l,m}(\omega,t) = \sum_{n\leq N}\lambda_{l,m}^n(\omega)cos\left(\frac{\pi nt}{T}\right)$$

with $\{\lambda_{l,m}^n\}$ denoting independent, uniformly distributed $U(-1,1)$ random variables. The geometry of the randomly evolving surface has now been described, and as a result the random normal velocity field $v_\nu^{\Gamma_\omega(t)}$ is determined. We consider a random physical tangential advection of material points over the surface $\Gamma_\omega(t)$, by

$$v_\tau^{\Gamma_\omega}\circ\phi = \mathcal{P}_{\Gamma_\omega(t)}\overline{v}(x),$$

for the smooth velocity field $\overline{v}(x) = x$, and we note the following given formula for the random unit normal vector field ν^{Γ_ω}, which we required in computations

$$\nu^{\Gamma_\omega}\circ\phi = \frac{\nu^{\Gamma_0} - A\nabla_{\Gamma_0}h}{|\nu^{\Gamma_0} - A\nabla_{\Gamma_0}h|} \qquad A = (I + h\mathcal{H})^{-1}.$$

The path-wise solution for the reformulated equation will be selected by

$$u(\omega,t,x) = s(t)sin\left(\frac{\pi}{2}(xy - r(t))\right)e^{1-z} + \sigma_{tol}\lambda(\omega)s(t)sin\left(\frac{\pi}{4}(x-z)\right)e^{y-1},$$

where $\sigma_{tol} > 0$ controls the maximum deviation of the pathwise solution from its mean, $\lambda \sim U(-1,1)$ is a uniformly distributed random variable, and where the above functions are defined as $r(t) = \frac{1}{4}(1-\frac{t}{T})$, $s(t) = exp\left(\frac{t^2}{t^2-K^2T^2}\right)$ with $K = 1.4$. The data f is then chosen such that u is precisely the pathwise solution of the reformulated equation. The numerical results are now as follows, where we have employed a Crank-Nicoloson time stepping in the L^2-estimates and an implicit Euler discretisation for the H^1-estimate, to obtain optimal convergence rates by scaling time stepping of the order h, i.e. $\tau \sim \mathcal{O}(h)$. Furthermore, an expansion of the order $N = 10$ is used in the fourier series representation of the random coefficients in time, and tolerances for the random height function and maximum deviation of the pathwise solution from its mean value are set as $\epsilon_{tol} = 0.1$, $\sigma_{tol} = 0.1$ respectively.

h	τ	M	$E_{L^\infty(0,T;L^2(\Omega^M;L^2(\Gamma_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.235702	0.125	1	0.8303	-	-	-
0.123091	0.0625	16	0.52744	0.698467	0.654627	-0.163657
0.0622573	0.03125	256	0.14108	1.93458	1.90249	-0.475624
0.0312195	0.015625	4096	0.0358821	1.98352	1.97517	-0.493793

Table 3.1: Error in $L^\infty(0,T;L^2(\Omega^M;L^2(\Gamma_0)))$.

h	τ	M	$E_{L^2(0,T;L^2(\Omega^M;H^1(\Gamma_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.235702	0.125	64	5.92613	-	-	-
0.123091	0.0625	256	4.82901	0.315143	0.295363	-0.147682
0.0622573	0.03125	1024	2.38627	1.03412	1.01697	-0.508485
0.0312195	0.015625	4096	1.20987	0.984045	0.979906	-0.489953

Table 3.2: Error in $L^2(0,T;L^2(\Omega^M;H^1(\Gamma_0)))$.

We conclude with some plots over time of a realisation of the path-wise solution u, pushed-forward onto the corresponding realisation of the randomly evolving surface $\Gamma_\omega(t)$. Furthermore, we provide plots of some of the random coefficients given in the spherical harmonic expansion of the random height function, for particular spherical harmonic functions.

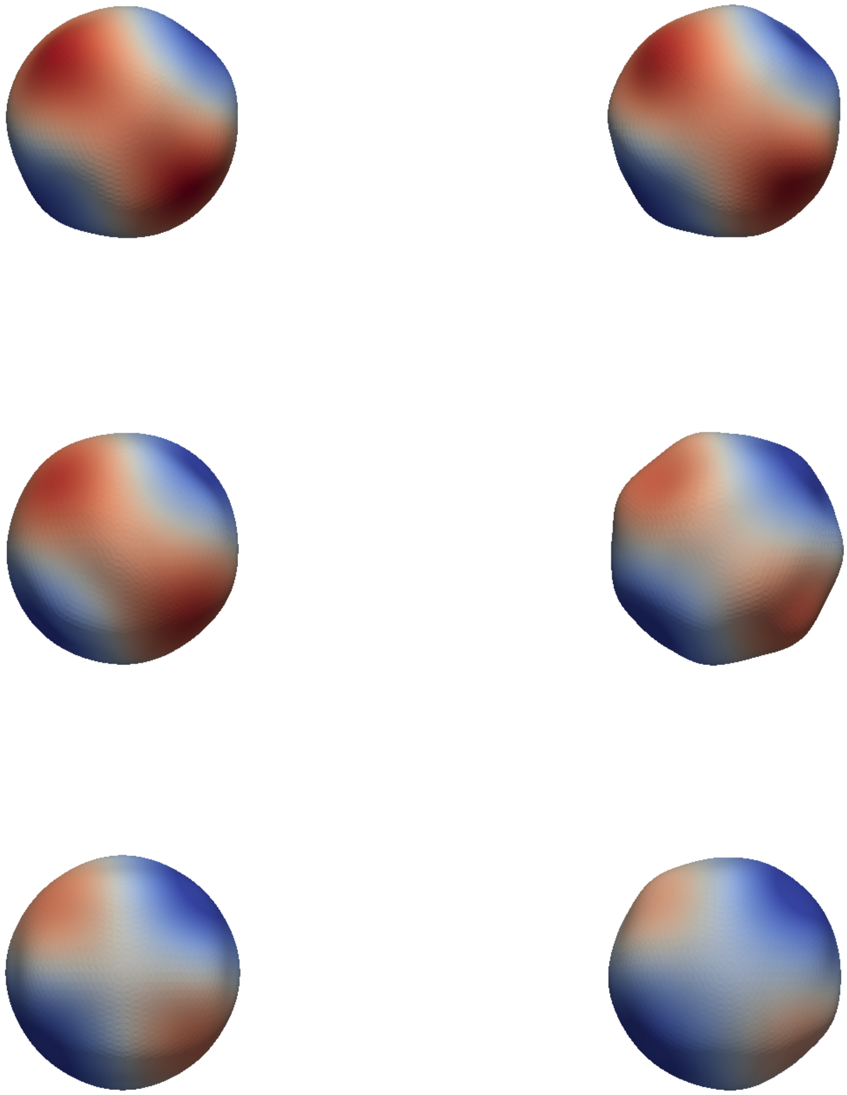

Figure 3.5: A realisation of the pathwise solution u on the corresponding realisation of the randomly evolving surface $\Gamma_\omega(t)$.

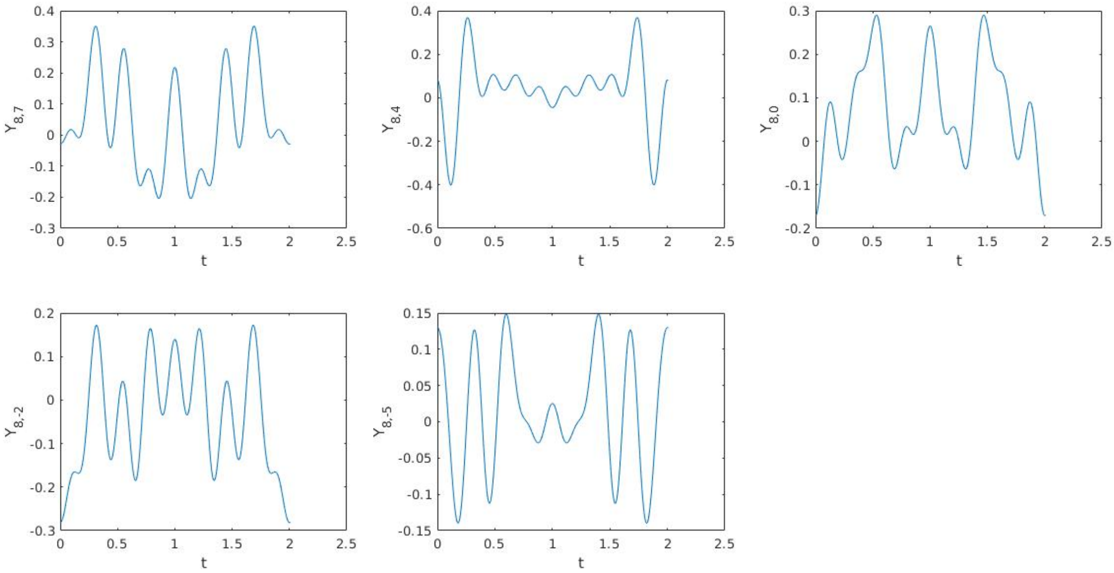

Figure 3.6: Plots of a realisation of some of the random coefficients over time given in expansion of the random height function, for particular spherical harmonics.

3.8.2 Advection-diffusion system on a randomly evolving bulk-surface

We next consider our second model problem, comprised of a coupled system of advection-diffusion equations on a randomly evolving bulk-surface. For our numerical example, we define the randomly evolving surface boundary $\Gamma_\omega(t)$, by a fluctuating graph

$$\Gamma_\omega(t) = \{x + h(\omega, t, x)\nu^{\Gamma_0}(x) \,|\, x \in \Gamma_0\}$$

over the unit circle $\Gamma_0 = S^1$, where the random height process is prescribed by a truncated fourier series expansion

$$h(\omega, t, x) = \sigma_{tol} \sum_{n \leq N_1} \lambda_n(\omega, t) cos(n\theta) + \hat{\lambda}_n(\omega, t) sin(n\theta) \qquad x = (cos\theta, sin\theta),$$

whose random coefficients are given by

$$\lambda_n(\omega, t) = \sum_{k \leq N_2} \lambda_{n,k}(\omega) cos\left(\frac{\pi k t}{T}\right)$$
$$\hat{\lambda}_n(\omega, t) = \sum_{k \leq N_2} \hat{\lambda}_{n,k}(\omega) cos\left(\frac{\pi k t}{T}\right)$$

for $\{\lambda_{n,k}, \hat{\lambda}_{n,k}\}$ independent, uniformly distributed $U(-1, 1)$ random variables. This completely describes the geometry of the randomly evolving bulk-surface $\overline{D_\omega(t)}$, and consequently the random normal velocity $w_\nu^{\Gamma_\omega}$ of the surface boundary $\Gamma_\omega(t)$. We consider a physical advection of material points in the domain by

$$w = \begin{cases} w_\tau^{\Gamma_\omega} + w_\tau^{\Gamma_\omega} & \text{on } \Gamma_\omega(t) \\ 0 & \text{in } D_\omega(t) \end{cases}$$

for the random tangential velocity field

$$w_\tau^{\Gamma_\omega} = \mathcal{P}_{\Gamma_\omega}\overline{v} \qquad \overline{v}(x) = x.$$

To apply the extended domain mapping method, we extend the boundary process into the interior domain with a blending function in the normal direction. More precisely, we define the blending function $B_\delta(\cdot) : [0,\infty) \to [0,\infty)$ by

$$B_\delta(x) = \begin{cases} exp\left(\frac{-x^2}{\delta^2 - x^2}\right) & x \leq \delta, \\ 0 & x \geq \delta, \end{cases}$$

where $\delta > 0$ is a parameter controlling the width of the neighbourhood in which the blending function is applied. The stochastic domain mapping is then defined by

$$\phi(\omega,t,x) = x + B_\delta\left(|x - a^{\Gamma_0}(x)|\right) h(\omega,t,a^{\Gamma_0}(x))\nu^{\Gamma_0}(a^{\Gamma_0}(x))) \qquad x \in \overline{B_1(0)},$$

and we may reformulate the system onto the reference bulk-surface, the closed unit disc $\overline{B_1(0)}$. We select a pathwise bulk solution of the reformulated system as

$$u(\omega,t,x) = sin(\pi xy)cos(\pi y^2) + \epsilon_{tol}\lambda(\omega)cos(\pi(y - \frac{t}{T}))sin(\pi x(1 - \frac{t}{T}))$$

and determine the surface pathwise solution via the reformulated Robin boundary condition

$$\alpha u - \beta v + \frac{\sqrt{g}}{\sqrt{g_{\Gamma_0}}} G^{-1}\nu^{\Gamma_0} \cdot \nabla u = 0 \qquad \text{on } \Gamma_0.$$

The data (f, f_{Γ_0}) is selected such that (u,v) solves the reformulated problem with the given data. The numerical results are as follows, where set the order of the fourier series expansion for the height function to be $N = 7$ and $N_2 = 10$ for the expansion of the coefficients in time. The tolerances are chosen by $\sigma_{tol} = 0.1$ and $\epsilon_{tol} = 0.1$. Furthermore, as with our previous example, we use a Crank-Nicoloson time-stepping for the L^2-estimates and an implicit Euler stepping for the H^1-estimates.

h	τ	M	$E_{L^\infty(0,T;L^2(\Omega^M;L^2(D_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.27735	0.125	1	0.619144	-	-	-
0.156174	0.0625	16	0.198298	1.98249	1.64261	-0.410651
0.0830455	0.03125	256	0.0540441	2.05828	1.87546	-0.468866
0.0428353	0.015625	4096	0.0136872	1.98131	1.98131	-0.495327

Table 3.3: Bulk error in $L^\infty(0,T;L^2(\Omega^M;L^2(D_0)))$.

h	τ	M	$E_{L^2(0,T;L^2(\Omega^M;H^1(D_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.27735	0.125	64	3.41133	-	-	-
0.156174	0.0625	256	2.17523	0.783494	0.649166	-0.324584
0.0830455	0.03125	1024	1.08874	1.09584	0.998508	-0.499252
0.0428353	0.015625	4096	0.55599	1.01511	0.969529	-0.484767

Table 3.4: Bulk error in $L^2(0,T;L^2(\Omega^M;H^1(D_0)))$.

h	τ	M	$E_{L^\infty(0,T;L^2(\Omega^M;L^2(\Gamma_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.27735	0.125	1	5.0787	-	-	-
0.156174	0.0625	16	1.06707	2.71654	2.25080	-0.562702
0.0830455	0.03125	256	0.28356	2.0983	1.91193	-0.477981
0.0428353	0.015625	4096	0.0723061	2.06414	1.97146	-0.492866

Table 3.5: Surface error in $L^\infty(0,T;L^2(\Omega^M;L^2(\Gamma_0)))$.

h	τ	M	$E_{L^2(0,T;L^2(\Omega^M;H^1(\Gamma_0)))}$	$eoc(h)$	$eoc(\tau)$	$eoc(M)$
0.27735	0.125	64	15.5792	-	-	-
0.156174	0.0625	256	7.85391	1.1926	0.98814	-0.494068
0.0830455	0.03125	1024	4.20041	0.990894	0.90288	-0.451441
0.0428353	0.015625	4096	2.12783	1.02727	0.98115	-0.490574

Table 3.6: Surface error in $L^2(0,T;L^2(\Omega^M;H^1(\Gamma_0)))$.

We conclude by plotting a realisation of the random bulk and surface quantity over time, pushed forward onto the associated realistion of the random bulk-surface.

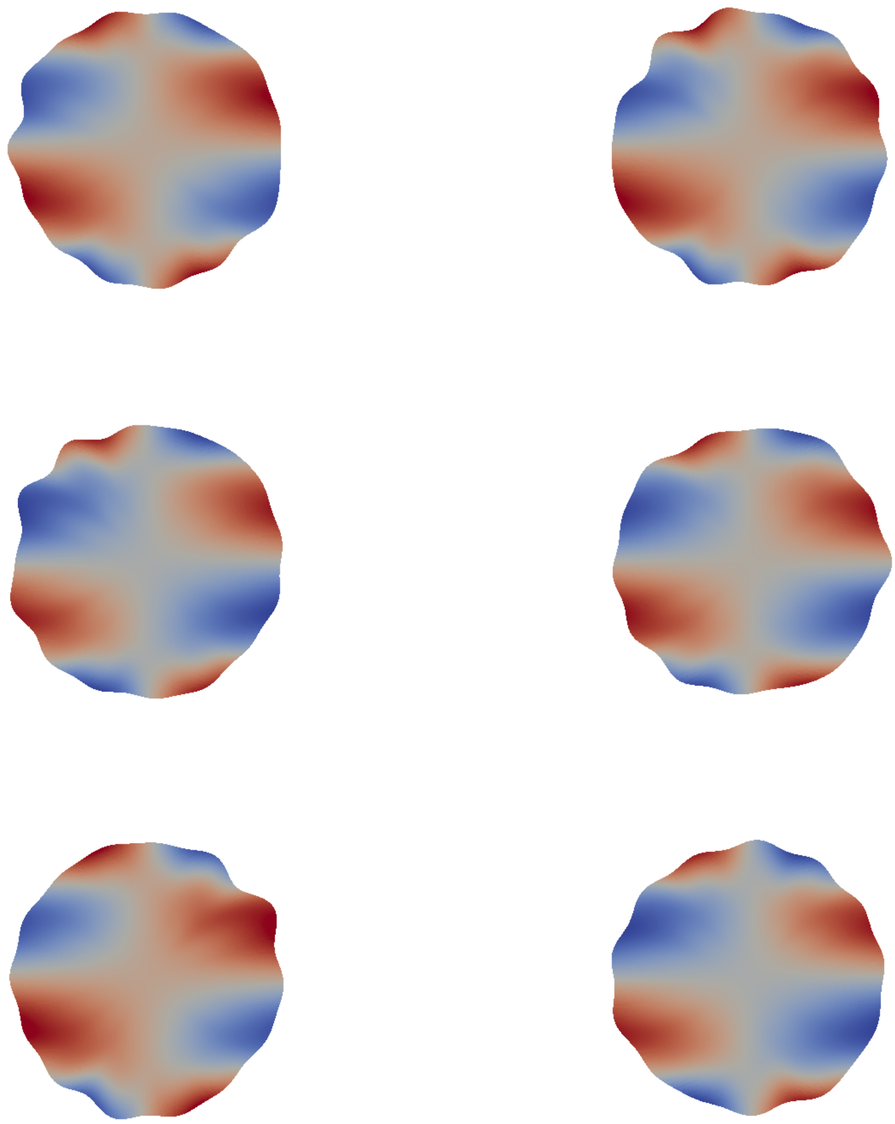

Figure 3.7: A realisation of the pathwise bulk-surface solution on the corresponding realisation of the randomly evolving bulk-surface.

Chapter 4

A future outlook on the applications of the domain mapping method

In this thesis, we have presented and analysed applications of the domain mapping method for advection-diffusion equations on randomly evolving surfaces and bulk-surface systems and furthermore their stationary counterparts. However, the scope for the domain mapping method extends far beyond the problems presented in this thesis. To illustrate but just of a few of the potential applications, we will consider the following two problems: a Hele-Shaw problem on a random surface and a two-phase Stefan problem on a random surface. These examples will hopefully demonstrate the flexibilty and potential that the domain mapping method has to offer, in incorporating and analysing geometric uncertainty into a range of different models.

4.1 A Hele-Shaw problem on a random surface

The Hele-Shaw problem, provides a model for the injection of a viscous fluid in a narrow gap between two parallel plates. The problem has been widely studied from an analytical and numerical perspective [37, 38, 89, 93], due to its wide applications in manufacturing and the resin injection moulding process [11, 41, 85]. The problem so far has been largely limited to the consideration of the injection process over a flat deterministic cell. However, with the domain mapping approach, an extension of the problem may be considered to incorporate potential geometric uncertainty which may naturally arise in the manufacturing process. We will now proceed by describing some of the main approaches to the deterministic case, which we have been adapted to the case of surface, and after which we will discuss how the problem may be extended to a random domain via the domain mapping method.

4.1.1 A variational inequality approach to the Hele-Shaw problem on a deterministic surface

Let $D \subset \mathbb{R}^3$ be a smooth, bounded surface with a boundary $\Gamma = \partial D$, that is partitioned into two disjoint sets $\Gamma = \Gamma_N \cup \Gamma_I$, which will respectively represent a region Γ_N in which a Neumann boundary condition will be imposed and a region Γ_I, in which the fluid is injected. For a given $0 < T < \infty$ fixed, the Hele-Shaw free boundary problem with fluid injection reads as follows:

Problem 4.1.1 (Hele-Shaw free boundary problem)**.** *Given $D(0)$ and $Q : (0,T] \to \mathbb{R}_{>0}$, determine $(p, D(t), \Gamma_f(t))$ for each $t \in (0,T]$ such that,*

$$-\nabla_D \cdot (k(x)\nabla_D p(x,t)) = 0 \quad \text{in } D(t), \tag{4.1.1}$$

$$p = 0 \quad \text{and} \quad k\nabla_D p \cdot \nu = -w \cdot \nu \quad \text{on } \Gamma_f(t) = \partial D(t) \cap D, \tag{4.1.2}$$

$$k\nabla_D p \cdot \nu = 0 \quad \text{on } \Gamma_N, \tag{4.1.3}$$

$$k\nabla_D p \cdot \nu = Q(t) \quad \text{on } \Gamma_I. \tag{4.1.4}$$

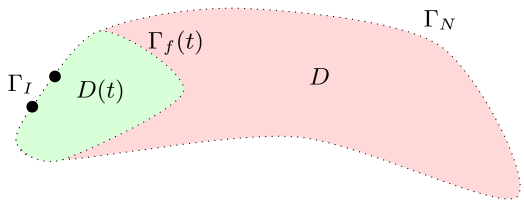

Figure 4.1: Fluid injection over the surface D

Here the unknown moving domain $D(t)$, the first phase, represents the region occupied by the fluid at the given time t. Its boundary $\Gamma_f(t)$, will evolve with the velocity field w, determined by the second condition in (4.1.2), also known as the jump condition. The outer unit normal vector to $\partial D(t)$ and ∂D, which is parallel to the surface D, will be denoted by ν. We will assume that the walls of the container where the fluid is not being injected Γ_N, are impervious to the flow and thus have the condition (4.1.3). Finally, we assume the coefficient k, is sufficiently smooth and satisfies

$$k(x) \geq k_0 > 0 \quad \text{for all } x \in D.$$

The above assumption ensures that the free boundary $\Gamma_f(t)$ advances over time, a property that can demonstrated using a maximum principle argument, as may be found in [93]. We will now provide a quick overview of the proof provided in [93], which we adapt to the case of a surface.

Lemma 4.1.1. *Let $(p, D(t), \Gamma_f(t))$ be a solution to the free boundary problem. Then for all $0 < t \leq t' \leq T$, we have*

$$p(\cdot,t) \geq 0 \quad \text{in } D(t), \quad D(t) \subseteq D(t'). \tag{4.1.5}$$

Proof. Let us multiply the elliptic equation (4.1.1) by a smooth test function $\phi \in H^1(D(t))$, integrate by parts and substitute in the boundary conditions (4.1.2 - 4.1.4) to obtain

$$\int_{D(t)} k\nabla_D p \cdot \nabla_D \phi - \int_{\Gamma_I} Q(t)\phi + \int_{\Gamma_f(t)} (w \cdot \nu)\phi = 0.$$

Choosing $\phi = p^- = \min(p, 0)$ and noting that the pressure is zero on the interface $\Gamma_f(t)$, we observe

$$\int_{D(t)} k\nabla_D p \cdot \nabla_D p^- = \int_{\Gamma_I} Q(t)p^- \leq 0.$$

It therefore follows that

$$k_0 \|\nabla_D p^-\|^2_{L^2(D(t))} \leq \int_{D(t)} k\nabla_D p \cdot \nabla_D p^- \leq 0,$$

and thus p^- is constant in $D(t)$. Since the $p = 0$ on $\Gamma_f(t) = \partial D(t)$, we conclude that $p^- \equiv 0$ in $D(t)$ and hence $p \geq 0$ in $D(t)$. The monotonicity of the evolving first phase $D(t)$, now follows from the jump condition (4.1.2) as the normal velocity $w \cdot \nu$ of the interface is non-negative. $\square$

We may therefore assume that there exists a function $l(x)$, for which we can represent the first phase $D(t)$ and the interface $\Gamma_f(t)$ at each time $t \in (0, T]$ by

$$D(t) = \{x \in D \,|\, l(x) < t\} \quad \Gamma_f(t) = \{x \in D \,|\, l(x) = t\},$$

where $l(x) = 0$ for all $x \in D_0$. With the above level set description of the interface which we may rewrite as $\Gamma_f(t) = \{x \in D \,|\, \phi(x,t) = l(x) - t = 0\}$, we can compute the following expressions for the outer unit normal to $D(t)$ and the normal velocity of the free boundary using the formulae $\nu = \frac{\nabla\phi}{|\nabla\phi|}$ and $w \cdot \nu = \frac{-\phi_t}{|\nabla_D\phi|}$

$$\nu = \frac{\nabla_D l}{|\nabla_D l|} \qquad w \cdot \nu = \frac{1}{|\nabla_D l|}.$$

Hence the evolution equation (4.1.2) for the free boundary $\Gamma_f(t)$ can equivalently be expressed as

$$k\nabla_D p \cdot \nabla_D l = -1. \tag{4.1.6}$$

Remark 4.1.1 (Final time condition)**.** *Naturally, the Hele-Shaw model breaks down after the fluid has completely filled the whole domain D. We will therefore now proceed by deriving an expression to determine the final completion time which we shall denote by T. Following a similar prodecure as found in the previous maximum principle argument, we begin by integrating the elliptic equation over $D(t)$ and integrating by parts to deduce*

$$0 = -\int_{D(t)} \nabla_D \cdot (k\nabla_D p) = -\int_{\Gamma_I} Q(t) + \int_{\Gamma_f(t)} w \cdot \nu.$$

Next, we note that with the transport property we have

$$\frac{d}{dt}\int_{D(t)} = \int_{D(t)} \nabla_D \cdot w = \int_{\Gamma_f(t)} w \cdot \nu,$$

and therefore deduce that the rate of change in the area of the fluid is precisely given by

$$\frac{d}{dt}\int_{D(t)} = \int_{\Gamma_I} Q(t).$$

Integrating in time then leads to the following condition to determine the final completion time

$$|D \setminus D_0| = \int_0^T \int_{\Gamma_I} Q(t)\, dt = |\Gamma_I| \int_0^T Q(t)\, dt. \tag{4.1.7}$$

We finally comment for later reference that

$$|D \setminus D_0| - |\Gamma_I| \int_0^t Q(s)\, ds = |\Gamma_I| \int_t^T Q(s) ds. \tag{4.1.8}$$

Following the approaches presented in [38, 93], we now proceed by deriving a variational inequality over D for the free boundary problem, by considering the following Baiocchi transformation

$$u(x,t) = \int_0^t p(x,t')dt' = \begin{cases} \int_{l(x)}^t p(x,t')dt' & x \in D(t) \\ 0 & x \in D \setminus D(t). \end{cases} \tag{4.1.9}$$

Here, we recall that the function $l(x)$ is defined as the first time at which the interface $\Gamma_f(t)$ touches the point x. To determine the elliptic differential equation which the new variable u satisfies, we begin by taking the tangential derivative of u at a point $x \in D(t)$, which if we differentiate under the integral we obtain

$$\nabla_D u(x,t) = \int_{l(x)}^t \nabla_D p(x,t')dt' - p(x, w(x))\nabla_D l(x) \qquad x \in D(t).$$

Since at the time $t = l(x)$, the point x lies on the interface, the pressure will be zero and so the last term vanishes. We can now calculate an expression for the elliptic differential operator in the first phase applied to the new variable u as

$$-\nabla_D \cdot (k(x)\nabla_D u(x,t)) = \int_{l(x)}^t -\nabla_D \cdot \left(k(x)\nabla_D p(x,t')\right) dt' + k(x)\nabla_D p(x, l(x)) \cdot \nabla_D l(x).$$

and therefore with jump condition (4.1.6), we arrive at the following free boundary problem.

Problem 4.1.2 (Free boundary problem for u)**.** *Given $D(0)$ and $Q : (0,T] \to \mathbb{R}_{>0}$, determine $(u, D(t), \Gamma_f(t))$ for each $t \in (0,T]$ such that,*

$$-\nabla_D \cdot (k(x)\nabla_D u(x,t)) = f(x) \quad \text{in } D(t), \tag{4.1.10}$$

$$u = 0 \quad \text{and} \quad k\nabla_D u \cdot \nu = 0 \quad \text{on } \Gamma_f(t) = \partial D(t) \cap D, \tag{4.1.11}$$

$$k\nabla_D u \cdot \nu = 0 \quad \text{on } \Gamma_N, \tag{4.1.12}$$

$$k\nabla_D u \cdot \nu = \mathcal{Q}(t) \quad \text{on } \Gamma_I. \tag{4.1.13}$$

Here $f = \chi_{D_0} - 1$ and $\mathcal{Q}(t) = \int_0^t Q(t')dt'$.

As a consequence of the pressure $p(\cdot, t)$ being non-negative in whole domain D and zero outside the first phase $D \setminus D(t)$, we may observe that the above free boundary problem for u is equivalent to the following linear complementarity problem posed over the whole domain D,

$$\begin{aligned}
-\nabla_D \cdot (k(x)\nabla_D u(x,t)) - f &\geq 0 \\
u &\geq 0 \\
(-\nabla_D \cdot (k(x)\nabla_D u(x,t)) - f) \cdot u &= 0
\end{aligned}$$

subject to boundary conditions (4.1.12, 4.1.13). Therefore to derive a variational inequality for u, we may define the closed, convex set

$$K = \{\phi \in H^1(D) \,|\, \phi \geq 0\}$$

and observe that for an arbitrary $v \in K$ we have

$$0 \leq \int_{D \backslash D(t)} (-\nabla_D \cdot (k\nabla_D u) - f)\,(v-u) = \int_D (-\nabla_D \cdot (k\nabla_D u) - f)\,(v-u).$$

Therefore if we define the bilinear form $a(\cdot,\cdot)$ and linear functional $l(t;\cdot)$ by

$$\begin{aligned}
a(u,v) &= \int_D k\nabla_D u \cdot \nabla_D v \\
l(t;v) &= \int_D fv + \int_{\Gamma_I} \mathcal{Q}(t)v,
\end{aligned}$$

we arrive at the following set of variational inequalities parametrised over time.

Problem 4.1.3 (Evolutionary variational inequality for u)**.** *For each $t \in (0,T)$, find $u(t) \in K$, such that for all $v \in K$, we have*

$$a(u(t), v - u(t)) \geq l(t; v - u(t)). \tag{4.1.14}$$

It is worth noting that these variational inequalities are equivalent to minimising the following functional

$$J(t;v) = \frac{1}{2}\int_D k|\nabla_D v|^2 - \int_D fv - \int_{\Gamma_I} \mathcal{Q}(t)v$$

over the set K at each time t. We may therefore employ standard optimisation results [65], to deduce the existence of a minimiser over K and thus a solution to (4.1.14). For this, it is sufficient to show that the functional $J(t;\cdot)$ is continuous, convex and coercive, i.e.

$$J(t;v) \to +\infty \quad \text{as } \|v\|_{H^1(D)} \to +\infty.$$

While the bilinear form $a(\cdot,\cdot)$ is only semi-coercive, the authors in [38] established coercivity of the functional $J(t;\cdot)$ at each time $t \in (0,T)$, by considering the following splitting of the space $H^1(D)$

$$v = Pv + Rv \qquad v \in H^1(D)$$

where

$$Pv := \frac{1}{|D|}\int_D v \qquad Rv := v - Pv.$$

With the above splitting, the functional may now be decomposed into the following components

$$J(t;v) = J(t;Rv) + J(t;Pv) = J(t;Rv) - \int_D f\,Pv - \int_{\Gamma_I} \mathcal{Q}(t)\,Pv,$$

which if we recall that $f = \chi_{D_0} - 1$ and $\mathcal{Q}(t) = \int_0^t Q(s)\,ds$ and furthermore substitute in the final time condition (4.1.8) leads to

$$\begin{aligned}
&= J(t;Rv) + Pv\left(\int_{D\setminus D_0} - |\Gamma_I|\int_0^t Q(s)\,ds\right)\\
&= J(t;Rv) + Pv\,|\Gamma_I|\int_t^T Q(s)\,ds.
\end{aligned}$$

Coercivity of $J(t;\cdot)$ may now be easily observed at each time $t \in (0,T)$, by applying Poincaré inequality to the zero-mean term Rv, and noting the non-negativity of the constant $|\Gamma_I|\int_t^T Q(s)$ as well as Pv since $v \in K$. We will now summarise these result below and further state an analagous regularity result for the surface as was established in the case of flat domain [38], for which a proof is required.

Theorem 4.1.1. *For each $t \in [0,T)$, there exists a unique solution to $u(t)$ to (4.1.14). Furthermore, $u(t) \in H^2(D)$ and there exists a constant $C > 0$ independent of $t \in [0,T)$ such that*

$$(T-t)|u(t)|_{L^2(D)} + |u(t)|_{H^1(D)} + |u(t)|_{H^2(D)} \le C,$$

where $|\cdot|_k$ denotes the relevant semi-norms.

4.1.2 A finite element approximation

We next outline a finite element approximation for the variational inequalities (4.1.14) based upon the approach presented in [37] for the flat case. We begin by approximating the smooth surface D by a polyhedral domain

$$D_h = \bigcup_{T\in\mathcal{T}_h} T$$

consisting of closed simplices T, whose vertices X_j are taken to lie on the smooth surface and whose maximum length is bounded by h. We impose the same conditions on our surface triangulation as was presented in Chapter 2. This induces a piecewise linear approximation of the boundary ∂D given by

$$\Gamma_h := \partial D_h = \bigcup_{\hat{T}\in\hat{\mathcal{T}}_h} \hat{T},$$

where $\hat{\mathcal{T}}_h$ denotes the associated triangulation of the boundary ∂D_h. We will assume that the interiors of the lifts of simplices $\hat{T}$ in our boundary triangulation under the close point projection mapping $a(\cdot)$, lie either completely within Γ_I or Γ_N, that is $\hat{T}^l \cap \Gamma_I = \emptyset$ or $\hat{T}^l \cap \Gamma_N = \emptyset$, and we will further denote the induced linear approximation of the region of fluid injection Γ_I by

$$\Gamma_{I,h} = \bigcup_{\substack{\hat{T} \in \hat{\mathcal{T}}_h \\ \hat{T}^l \cap \Gamma_I \neq \emptyset}} \hat{T}.$$

We next define a linear finite element space on the triangulation by

$$S_h = \{\phi_h \in C^0(D_h) \,|\, \phi_h|_T \text{ is affine linear for all } T \in \mathcal{T}_h\}$$

and define an approximation of the closed convex set $K = \{\phi \in H^1(D) \,|\, \phi \geq 0\}$ by

$$K_h = \{\phi_h \in S_h \,|\, \phi_h \geq 0\}.$$

We then approximate the continuous problem with the following discrete bilinear form and linear functional

$$\begin{aligned} a_h(W_h, V_h) &= \int_{D_h} k^{-l} \nabla_{D_h} W_h \cdot \nabla_{D_h} V_h, \\ l_h(t; V_h) &= \int_{D_h} f_h V_h + \int_{\Gamma_{I,h}} \mathcal{Q}(t) V_h, \end{aligned}$$

where $f_h := \chi_{D_{0,h}}$, with $D_{0,h}$ denoting our approximation of the initial domain occupied by the fluid D_0. Our approximation of the variational inequalities is then as follows.

Problem 4.1.4 (Finite element approximation)**.** *For each $t \in (0, T)$, find $U_h(t) \in K_h$, such that for all $V_h \in K_h$ we have*

$$a_h(U_h(t), V_h - U_h(t)) \geq l_h(t; V_h - U_h(t)). \tag{4.1.15}$$

The following error estimate was derived in [37] in the case of a flat domain, which we have now restated in terms of the deterministic surface. A proof of the analgous error bound result stated below is required. Note that here we have denoted by u_h, the lift of the discrete solution U_h under the closest point projection mapping $a(\cdot)$.

Theorem 4.1.2 (Error estimate)**.** *There exists a constant $c > 0$ such that for all $t \in [0, T_h^*)$, we have*

$$\|u(t) - u_h(t)\|_{H^1(D)} < ch/(T_h^* - t)^2,$$

where $T_h^* = \min(T, T_h)$, with the discrete final time T_h determined by

$$|D_h \setminus D_{h,0}| = |\Gamma_{I,h}| \int_0^{T_h} Q(t)\, dt.$$

4.1.3 Numerical computation

In this section, we will outline the linear complementarity problems (LCP) which arise from the discrete finite element approximations of the variational inequalities, and a discuss a numerical method, the projected SOR method, that was first introduced in [22] to subsequently solve the equivalent LCPs. We begin by denoting the nodal basis of the finite element space S_h by ϕ_i, for $i = 1, ..., N$. If we express the discrete solution $U_h(t)$, and the difference $V_h - U_h(t)$ for an arbitrary $V_h \in S_h$ as

$$U_h(t) = \sum_{j=1}^{N} U_j(t)\phi_j \quad V_h - U_h(t) = \sum_{k=1}^{N} (V_k - U_k(t))\phi_k,$$

then we may equivalently rewrite the discrete problem (4.1.15) as the following system of algebraic inequalities

$$AU(t) \cdot (V - U(t)) \geq F(t) \cdot (V - U(t))$$

where $U(t) = (U_1(t), ...U_N(t))^\top$ and the coefficients $A = \{A_{ij}\}_{i,j=1,...,N}$ and $F(t) = \{F_j(t)\}_{j=1,...,N}$ are given by

$$A_{ij} = a_h(\phi_i, \phi_j) \qquad F_j(t) = l_h(t; \phi_j).$$

Taking $V = U(t) \pm \epsilon e_i$, in the case where $U_i > 0$ for an $\epsilon > 0$ sufficiently small, leads to the following linear complementarity problem:

Problem 4.1.5 (Linear complementarity problem). *For each $t \in (0, T)$, find $U(t) \in \mathbb{R}^N$ such that*

$$AU(t) - F(t) \geq 0 \tag{4.1.16}$$

$$U(t) \geq 0 \tag{4.1.17}$$

$$(AU(t) - F(t)) \cdot U(t) = 0. \tag{4.1.18}$$

We may now solve the above discrete problem in numerical computations via the projected SOR method, which is an iterative scheme based upon the usual SOR method to solve algebraic equations, but with an additional step at each stage in the iteration process, of imposing the non-negative constraints. For further details on the projected SOR method as a well convergence results, we refer the reader to [15, 20, 21, 71].

Algorithm 1 (The projected SOR method). *An approximation of $U(t)$ at a given $t \in (0, T)$.*
Step 1 *Choose an initial guess $U^0 \in \mathbb{R}^N$ with $U^0 \geq 0$ and a relation parameter $0 < \omega < 2$.*
Step 2 *Generate a sequence $U^n \in \mathbb{R}^N$ of approximation of $U(t)$ by sequentially updating the*

components of U_i^n *for* $i = 1, ..., N$ *by the following process*

$$\hat{U}_i^{n+1} = \frac{1}{A_{ii}} \left(F_i(t) - \sum_{j<i} A_{ij} U_j^{n+1} - \sum_{j>i} A_{ij} U_j^n \right),$$
$$U_i^{n+1} = \max\left(0, U_i^n + \omega(\hat{U}_i^{n+1} - U_i^n)\right).$$

As a motivating example for the Hele-Shaw flow over a deterministic surface, we have the following numerical simulation for a variable injection rate prescribed by $Q(t) = 1 + \lambda t^2$, with $\lambda = 0.5$. Here for simplicity k was taken to be constant $k \equiv 2$ but may in future be modified. The numerical results are as follows. It is worth emphasising that the numerical computation at a given time t does not require any prior knowledge of the numerical solution at previous times.

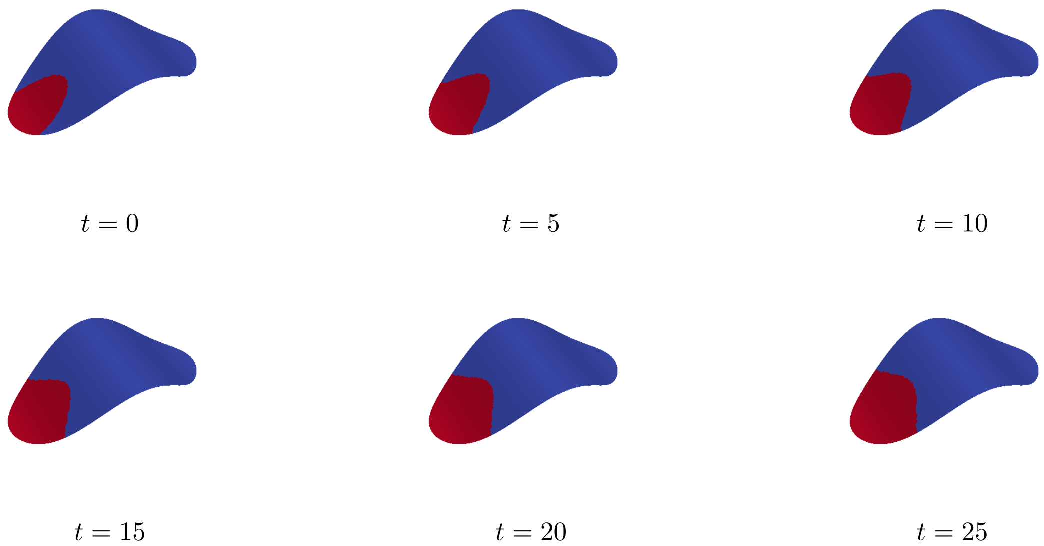

Figure 4.2: The flow of a fluid over time with a variable rate of injection.

The task now, and one of many possible future outlooks for the domain mapping method, is to consider the Hele-Shaw problem over a random surface. We will now continue by first discussing a possible formulation of this problem, including describing some of the details for the reformulation onto the reference domain, and second, we will discuss possible numerical approaches to effectively treat the randomness present in the model. Note that as our formulation we will rely upon a vector-valued random field representation of the random surface, we will now introduce here, the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω denoting the set of all elementary events for our domain mapping.

4.1.4 A Hele-Shaw problem on a random surface

The random surface container $D(\omega)$, in which the fluid is injected into, will be prescribed by

$$D(\omega) = \{\phi(\omega, x) \,|\, x \in D_{\text{ref}}\}$$

for a given vector-valued random field $\phi : \Omega \times \overline{D_{\text{ref}}} \to \mathbb{R}^3$, defined over a smooth deterministic reference surface D_{ref}. See figure 4.3 for a depiction of the geometric setting. We will assume that realisations of the domain mapping $\phi(\omega, \cdot)$ are sufficiently smooth to ensure a minimum regularity of the random surface $D(\omega)$, that will be required in our analysis of the Hele-Shaw problem. We will furthermore assume that the restriction of the domain mapping to the boundary ∂D_{ref}, maps onto the random boundary process

$$\phi(\omega, \cdot)|_{\partial D_{ref}} : \partial D_{ref} \to \partial D(\omega)$$

and furthermore that the random region of fluid injection $\Gamma_I(\omega)$ and impervious walls $\Gamma_N(\omega)$ of the container $\overline{D(\omega)}$, are characterised by

$$\Gamma_N(\omega) = \{\phi(\omega, x) \,|\, x \in \Gamma_{N,\text{ref}}\} \qquad \Gamma_I(\omega) = \{\phi(\omega, x) \,|\, x \in \Gamma_{I,\text{ref}}\} \tag{4.1.19}$$

for a disjoint partitioning of the reference boundary into two deterministic sets

$$\partial D_{ref} = \Gamma_{I,ref} \cup \Gamma_{N,ref}.$$

We may therefore consider in our reformulation of the Hele-Shaw problem onto the reference domain $\overline{D_{ref}}$, that $\Gamma_{I,\text{ref}}$ is representative of the region of fluid injection and $\Gamma_{N,ref}$ is the region of impervious walls. The rate at which fluid will be injected into the random surface container will be prescribed by $Q : [0, T] \to \mathbb{R}_{\geq 0}$ and therefore is taken to be deterministic. It is worth at this stage clarifying that the only sources of randomness considered in our model so far, are the geometric uncertainties in: the domain $D(\omega)$, the region of fluid injection $\Gamma_I(\omega)$, the regions of impervious walls $\Gamma_N(\omega)$, and the initial region occupied by the fluid $D_\omega(0)$, as all of these quantities will naturally depend on the realisation of the random surface $D(\omega)$. For convenience, we will impose of the following assumptions on the measures of these domains

$$|\Gamma_I(\omega)| = |\Gamma_{I,\text{ref}}| \qquad |D(\omega)| = |D_{\text{ref}}| \qquad |D_\omega(0)| = |D_{\text{ref}}(0)| \qquad \text{for a.e. } \omega.$$

This ensures that the final time taken $T(\omega)$ to completely fill the corresponding realisation of the random surface $D(\omega)$, which is given by the condition

$$|D(\omega) \setminus D_0(\omega)| = |\Gamma_I(\omega)| \int_0^T Q(s)ds, \tag{4.1.20}$$

is independent of the realisation taken. We will denote the final completion time for all realisations by T. These conditions are imposed only for convenience and may instead be omitted

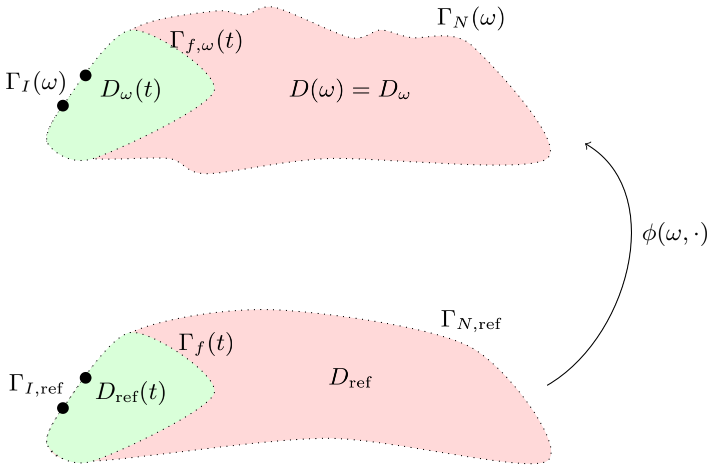

Figure 4.3: A realisation of a Hele-Shaw flow over a random surface D_ω and the corresponding pull-back onto the reference surface D_{ref}.

by alternatively setting the final time in the model as $T = \inf_\omega T(\omega)$. We will finally treat the prescribed diffusion coefficient k to be random and modelled as a random field $k(\omega, x)$. This leads to our following model for the Hele-Shaw problem posed over a random surface.

Problem 4.1.6 (Hele-Shaw free boundary problem on a random surface)**.** *Given $D_\omega(0)$ and $Q : (0, T] \to \mathbb{R}_{>0}$, determine $(p, D_\omega(t), \Gamma_{f,\omega}(t))$ for each $t \in (0, T]$ such that,*

$$-\nabla_{D_\omega} \cdot (k(\omega, x)\nabla_{D_\omega} p(\omega, x, t)) = 0 \quad \text{in } D_\omega(t), \tag{4.1.21}$$

$$p = 0 \quad \text{and} \quad k\nabla_{D_\omega} p \cdot \nu = -w \cdot \nu \quad \text{on } \Gamma_{f,\omega}(t) = \partial D_\omega(t) \cap D_\omega, \tag{4.1.22}$$

$$k\nabla_{D_\omega} p \cdot \nu = 0 \quad \text{on } \Gamma_N(\omega), \tag{4.1.23}$$

$$k\nabla_{D_\omega} p \cdot \nu = Q(t) \quad \text{on } \Gamma_I(\omega). \tag{4.1.24}$$

Employing the Baoicchi transformation by integrating realisations of the pressure $p(\omega, x, \cdot)$ in time

$$u(\omega, x, t) = \int_0^t p(\omega, x, t')dt' \qquad x \in D(\omega), \tag{4.1.25}$$

and following the analagous calculations presented for the deterministic case, leads us to the following set of stochastic elliptic variational inequalities with realisations posed over the closed convex set

$$K(\omega) = \{\phi \in H^1(D_\omega) \,|\, \phi \geq 0\}.$$

Problem 4.1.7 (Stochastic evolutionary variational inequality for u)**.** *For each $t \in (0, T)$ and a.e. ω, find $u(\omega, \cdot, t) \in K(\omega)$, such that for all $v \in K(\omega)$, we have*

$$a(\omega; u(\omega, t), v - u(\omega, t)) \geq l(\omega, t; v - u(\omega, t)). \tag{4.1.26}$$

Here, the realisations of the stochastic bilinear form $a(\omega; \cdot, \cdot)$ and linear functional $l(\omega, t; \cdot)$ are *defined for* $u, v \in H^1(D(\omega))$ *by*

$$a(\omega; u, v) = \int_{D_\omega} k(\omega)\nabla_{D_\omega} u \cdot \nabla_{D_\omega} v$$
$$l(\omega, t; v) = \int_{D_\omega} f(\omega)v + \int_{\Gamma_I(\omega)} \mathcal{Q}(t)v.$$

For convenience, we have suppressed parameters in the above definition, i.e. $k(\omega) = k(\omega, \cdot)$.

We may now employ the domain mapping method to reformulate realisations of the stochastic variational inequalities onto the deterministic reference surface D_{ref}. In order to pull-back the random boundary line integral over the injection region $\Gamma_{I,\text{ref}}(\omega)$, we have the following calculation.

Lemma 4.1.2. *Let* $\Gamma_I(\omega)$ *and* $\Gamma_{I,ref}$, *by as previously described in (4.1.19). Then for any* $f : \Gamma_I(\omega) \to \mathbb{R}$, *we have*

$$\int_{\Gamma_I(\omega)} f(y)\, ds(y) = \int_{\Gamma_{I,ref}} (f \circ \phi)(\omega, x)\, \sqrt{g_{\Gamma_I}(\omega, x)}\, ds(x) \tag{4.1.27}$$

where

$$\sqrt{g_{\Gamma_I}(\omega, x)} = |\nabla_{D_{ref}}\phi(\omega, x)\, \mu(x)|$$

with $\mu(x)$ *denoting the unique unit vector (up to sign) in the tangent space* $T_x D_{ref}$ *which is also tangent to the boundary* $\Gamma_{I,ref}$.

Proof. Let $r : [a, b] \to \mathbb{R}^3$ denote a parametrisation of the curve $\Gamma_{I,\text{ref}}$, which represents the region of fluid injection in the reference domain D_{ref}. It follows that if we denote realisations of the domain mapping via $\phi_\omega(\cdot) = \phi(\omega, \cdot)$, that the composite mapping

$$\phi_\omega \circ r : [a, b] \to \mathbb{R}^3$$

parametrises the corresponding realisation of the random region of fluid injection $\Gamma_I(\omega)$ on the random surface $D(\omega)$. Therefore, we may express the line integral in (4.1.27) as

$$\int_{\Gamma_I(\omega)} f(y)\, ds(y) = \int_a^b (f \circ \phi_\omega \circ r)(t)\, |(\phi_\omega \circ r)'(t)|\, dt.$$

Applying the chain rule, and noting that $r' \in TD_{ref}$, gives

$$(\phi_\omega \circ r)'(t) = \nabla_{D_{ref}}\phi_\omega(r(t))\, r'(t)$$

and therefore if we express $r'(t) = \frac{r'(t)}{|r'(t)|}\,|r'(t)|$ and observe that $\mu(r(t)) = \pm\frac{r'(t)}{|r'(t)|}$, we obtain the stated result

$$\int_{\Gamma_I(\omega)} f(y)\,ds(y) = \int_a^b (f \circ \phi_\omega \circ r)(t)\,|\nabla_{D_{ref}}\phi_\omega(r(t))\,\mu(r(t))|\,|r'(t)|\,dt \tag{4.1.28}$$

$$= \int_{\Gamma_{I,\mathrm{ref}}} f(\phi_\omega(x))\,|\nabla_{D_{ref}}\phi_\omega(x)\,\mu(x)|\,ds(x). \tag{4.1.29}$$

□

Remark 4.1.2. *(Area element for pull-back of line integrals via the domain mapping.) Note that we have chosen to adopt a similar notation for the measure of the area element* $\sqrt{g_{\Gamma_{I,ref}}}$ *corresponding to the pull-back of line integrals onto the reference domain via the domain mapping* $\phi(\omega,\cdot)$*, as was adopted for the pull-back of surface integrals considered in Chapter 2. To recap briefly, the area element corresponding to the pull-back of a surface integral onto the reference surface was denoted and prescribed by* $\sqrt{g_{D_{ref}}} = \sqrt{\det(G_{D_{ref}})}$*, where the matrix* $G_{D_{ref}} \in \mathbb{R}^{3\times 3}$ *is defined as*

$$G_{D_{ref}} = \nabla_{D_{ref}}\phi^\top \nabla_{D_{ref}}\phi + \nu \otimes \nu,$$

with ν *denoting the unit normal vector to the surface* D_{ref}*. The connection between the two area elements is given as through the restriction of the linear mapping* $G_{D_{ref}}$ *to the 1-dimensional linear subspace tangent to the boundary* ∂D_{ref}*, i.e.*

$$g_{\Gamma_{I,ref}} = G_{D_{ref}}\mu \cdot \mu. \tag{4.1.30}$$

We may now reformulate realisations of the stochastic variational inequality (4.1.26) posed over the random surface $D(\omega)$, onto the reference surface D_{ref}, using the previous calculation for the boundary integral term (4.1.28) and our general formulae for the pull-back of tangential derivatives derived in (2.3.12). For this, we first introduce the notation of a pull-back of a given function $u(\omega,\cdot) : D(\omega) \to \mathbb{R}$ onto the reference surface by

$$\hat{u}(\omega, x) = u(\omega, \phi(\omega, x)) \qquad x \in D_{ref}.$$

This leads to the consideration of the following stochastic elliptic variational inequality where the stochastic bilinear form $\hat{a}(\omega : \cdot, \cdot)$ and linear functional $\hat{l}(\omega, t; \cdot)$ are defined by

$$\hat{a}(\omega; \hat{u}, \hat{v}) = \int_{D_{ref}} \hat{k}(\omega)\sqrt{g_{D_{ref}}(\omega)}\,G^{-1}_{D_{ref}}(\omega)\nabla_{D_{ref}}\hat{u} \cdot \nabla_{D_{ref}}\hat{v}$$

and

$$\hat{l}(\omega, t; v) = \int_{D_{ref}} \hat{f}(\omega)\hat{v}\sqrt{g_{D_{ref}}(\omega)} + \int_{\Gamma_{I,ref}} \mathcal{Q}(t)\hat{v}\sqrt{g_{\Gamma_{I,ref}}(\omega)}$$

for $\hat{u}, \hat{v} \in H^1(D_{ref})$, where the random coefficient are given by

$$\begin{aligned} G_{D_{ref}}(\omega) &= \nabla_{D_{ref}}\phi(\omega)^\top \nabla_{D_{ref}}\phi(\omega) + \nu \otimes \nu, \\ g_{D_{ref}}(\omega) &= \det(G_{D_{ref}}(\omega)) \\ g_{\Gamma_{I,ref}}(\omega) &= \det(G_{D_{ref}}(\omega)\mu \cdot \mu) \end{aligned}$$

and the associated pull-backs of the random fields are denoted by $k \circ \phi(\omega) = \hat{k}(\omega)$ and $f \circ \phi(\omega) = \hat{f}(\omega)$.

Problem 4.1.8 (Stochastic evolutionary variational inequality for $\hat{u}$)**.** *For each $t \in (0, T)$ and a.e. ω, find $\hat{u}(\omega, \cdot, t) \in \hat{K} := \{\phi \in H^1(D_{ref}) \,|\, \phi \geq 0\}$, such that*

$$\hat{a}(\omega; \hat{u}(\omega, t), \hat{v} - \hat{u}(\omega, t)) \geq \hat{l}(\omega, t; \hat{v} - \hat{u}(\omega, t)) \tag{4.1.31}$$

for all $v \in \hat{K}$.

The challenges which remain ahead for future work, entail first providing suitable assumptions on the domain mapping $\phi(\omega, \cdot)$, which describes the random surface $D(\omega)$, and proving the existence and uniqueness of pathwise solutions to the stochastic variational inequality (4.1.31), following analagous arguments to those presented in [38, 93], for the deterministic case. This will include deriving finite second moments of the pathwise solution, through the careful treatment of all of the constants arising in the analysis and by employing the uniform bounds imposed on the domain mapping. After which, we will need to establish a higher regularity for the realisations of the solution, which will naturally follow from existing results in the deterministic case [17, 38, 90]. Uniform bounds on higher regularity estimates of the solution, which are required for our finite element error estimates, will also have to be derived following a similar careful treatment of all constants arising. The second challenge posed by the consideration of the formulation on a random surface, will entail the analysis of a numerical scheme. We have already described numerical results and error bounds for a finite element discretisation of the deterministic case, using piecewise linear elements, which results in solving a linear complementarity problem. This will now need to be combine with a discretisation of the stochastic variable, such as by a Monte-Carlo sampling method considered in our previous problems, and an error bound for the combined approach will be required. Consideration of a multi-level Monte-Carlo approach may also wished to be considered after their proposal in [59], in which the authors established a reduced computational cost, for a stochastic elliptical obstacle problem of the form (4.1.31).

4.2 A two-phase Stefan problem on a random surface

The two-phase Stefan problem, first introduced by Stefan [92] in 1890, provides a model for the heat transfer in a substance which undergoes a phase change. The model has a wide range of applications; in geophysics [72, 79] and crystalline growth [14, 58, 91], and consequently its analysis and numerical treatment has been widely analysed [33, 73, 77]. A key feature of the problem is the separation of the two phases by an unknown free boundary, whose evolution is prescribed via the Stefan condition. One of the proposed approaches to numerically treat the free boundary problem, is to approximate the solution of the heat equation in each of the two phases, whilst simultaneously tracking the front [64, 73, 74]. The accuracy of this approach relies heavily upon the approximation of the free boundary, and whilst it may offer a high level of accuracy, this is at the expense of a high computational cost [96]. In contrast to this approach, a reformulation of the free boundary problem in terms of the enthalpy has also been proposed in [33, 77, 99]. The enthalpy reformulation has the advantage of not needing to track the front, and instead leads to a consideration of a nonlinear parabolic equation, where the nonlinearity arises due to the discontinuous jump in the enthalpy across the free boundary. An analysis and the numerical treatment of the enthalpy reformulation have been considered in [33, 77, 99].

We will now proceed by providing a brief account of the analysis and numerical approximation of the enthalpy formulation, adapted to the case of a deterministic surface. These results would have to verified using analagous arguments as to those presented and subsequently referred to, which consider a flat domain. After which, we will outline the next challenge of incorporating geometric uncertainty into our model for the surface in which the substance resides over. In particular, we will discuss how the problem may be formulated and subsequently treated with the domain mapping method.

4.2.1 A Stefan problem on a deterministic surface

Let D denote a smooth compact surface in $\mathbb{R}^3$ and $u(x,t)$ the temperature of a given substance which resides over the surface. The substance under consideration will be assumed to undergo a phase transition in the region where $u(x,t)=0$. As a motivating example, and based upon the initial formulation of the Stefan problem [92], we may consider $u(x,t)$ to measure the temperature of regions of ice and water which partition the surface D. We will denote the corresponding domains for the two phases by

$$D_s(t) = \{x \in D \,|\, u(x,t) < 0\} \qquad D_l(t) = \{x \in D \,|\, u(x,t) > 0\}$$

and the interface between the phases via

$$\Gamma(t) = \{x \in D \,|\, u(x,t) = 0\}.$$

The two-phase Stefan problem formulated over the surface D subsequently reads as follows.

Problem 4.2.1. *Find $u : D \times (0,T] \to \mathbb{R}$ such that*

$$u_t - \Delta_D u = f \quad \text{in } D_s(t) \cup D_l(t) \tag{4.2.1}$$

$$u = 0, \quad -(\nabla_D u^l - \nabla_D u^s) \cdot \mu = V \quad \text{on } \Gamma(t) \tag{4.2.2}$$

$$u(0) = u_0 \quad \text{on } D. \tag{4.2.3}$$

Here $\mu(x,t)$ denotes the conormal vector to $D_s(t)$, that is the unique unit vector which is tangent to the surface D, orthogonal to the free boundary $\Gamma(t)$ and oriented in the direction of $D_l(t)$. The second condition in (4.2.2), also known as the Stefan condition, will determine the evolution of the free boundary $\Gamma(t)$ by prescribing its conormal velocity V. Here we have denoted by $\nabla_D u^l$ and $\nabla_D u^s$, the trace of the tangential derivative of the temperature u restricted to each of the respective domains $D_l(t)$ and $D_s(t)$, i.e. $\nabla_D u^l = tr(\nabla_D\, u|_{D_l(t)})$.

As previously discussed, we will evade the task of tracking the evolution of the front $\Gamma(t)$, by instead considering an enthalpy formulation of the free boundary problem following the approaches presented in [33, 77, 98]. Here, it is worth recalling that enthalpy, which we will denote by $e(x,t)$, measures the combined energy required to maintain the current state of the substance as well as its thermal energy. The quantity of energy required to change the state of the substance from solid to liquid, while maintaining a constant temperature is precribed by the latent heat of fusion $\lambda > 0$. The connection between the enthalpy and temperature of the considered substance is given by

$$\beta(e) = \begin{cases} \frac{1}{c_s}e & \text{for } e < 0 \\ 0 & \text{for } e \in [0, L] \\ \frac{1}{c_l}(e - L) & \text{for } e > 0, \end{cases}$$

where the constants $c_l > 0$ and $c_s > 0$ respectively respresent the specific heat in the liquid and solid states. For convenience, we shall set these constants as $c_s = c_l = 1$. The inverse relationship between the temperature and the enthalpy will subsequently be prescribed by the graph $H(\cdot)$, defined as $H(\cdot) = \beta(\cdot)^{-1}$, see figure 4.4. Here it is worth emphasising the discontinuous jump in the enthalpy over the interface $\Gamma(t)$, a feature which will form the crux of the challenge for the reformulated problem.

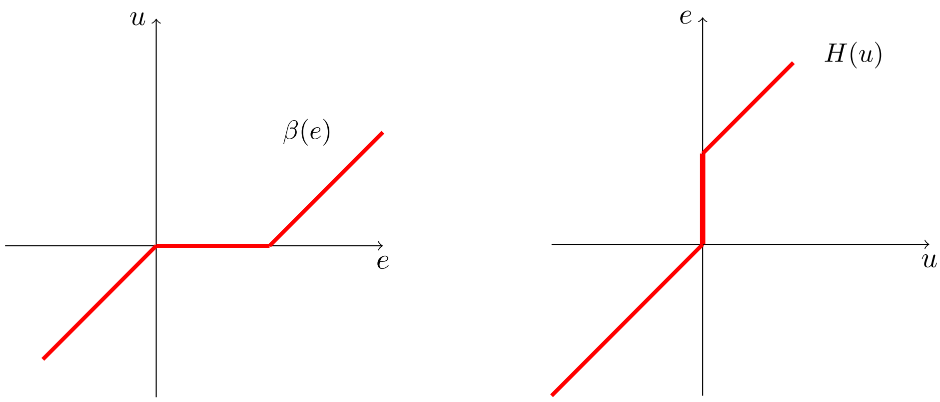

Figure 4.4: Graphs of the relationships between the enthalpy and temperature of the considered substance.

We now proceed by deriving a weak-formulation for the enthalpy following the derivation presented in [69]. We begin by multiplying the Laplacian term in $D_s(t)$ by a sufficiently smooth function $\phi(x,t)$ and integrating by parts to obtain

$$\int_{D_s(t)} -\Delta_D u\,\phi = \int_{D_s(t)} \nabla_D u \cdot \nabla_D \phi - \int_{\Gamma(t)} \phi \nabla_D u^s \cdot \mu. \tag{4.2.4}$$

A similar calculation in the liquid phase $D_l(t)$, leads to

$$\int_{D_l(t)} -\Delta_D u\,\phi = \int_{D_l(t)} \nabla_D u \cdot \nabla_D \phi + \int_{\Gamma(t)} \phi \nabla_D u^l \cdot \mu \tag{4.2.5}$$

where we observe a change of sign in the boundary term due to the orientation of $\mu(x,t)$. We next consider the time derivative of the integral of the enthalpy over the evolving solid phase $D_s(t)$, which with an application of the transport property gives

$$\frac{d}{dt}\int_{D_s(t)} e\phi = \int_{D_s(t)} (e\phi)_t + \int_{\Gamma(t)} e^s \phi V.$$

Here we recall that V denotes the conormal velocity of the free boundary $\Gamma(t)$ and furthermore that e^s denotes the trace of the enthalpy $e|_{D_s(t)}$ restricted to the domain $D_s(t)$. We may observe in figure 4.4, that as we approach the interface from within the region of the solid state, that the limit of the enthalpy is zero and so $e^s = 0$ and consequently the boundary term vanishes. Furthermore, we may also observe that $u_t = e_t$ in $D_s(t)$ and therefore deduce

$$\frac{d}{dt}\int_{D_s(t)} e\phi = \int_{D_s(t)} u_t\phi + e\phi_t. \tag{4.2.6}$$

By a similar argument in the evolving liquid phase $D_l(t)$, we have

$$\frac{d}{dt}\int_{D_s(t)} e\phi = \int_{D_s(t)} (e\phi)_t - \int_{\Gamma(t)} e^l\phi V = \int_{D_s(t)} u_t\phi + e\phi_t - \int_{\Gamma(t)} \phi V. \tag{4.2.7}$$

where in constrast for this particular case, we have that the trace of the enthalpy from the liquid phase will be $e^l = 1$ on $\Gamma(t)$. Combining (4.2.4) - (4.2.7) leads to

$$\int_D (u_t - \Delta_D u)\phi = \frac{d}{dt}\int_D e\phi - \int_D e\,\phi_t - \int_{\Gamma(t)} \phi V + \int_D \nabla_D u \cdot \nabla_D \phi - \int_{\Gamma(t)} \phi(\nabla_D u^l - \nabla_D u^s)\cdot \mu$$

and therefore if we substitute in the Stefan condition $(\nabla_D u^l - \nabla_D u^s)\cdot\mu = -V$, we observe that the free boundary terms cancel. Hence with the PDE satisfied pointwise by u, we arrive at

$$\int_D f\phi = \frac{d}{dt}\int_D e\phi - \int_D e\,\phi_t + \int_D \nabla_D u \cdot \nabla_D \phi$$

and thus the following weak formulation, which is a surface analogue of the formulations that may be found in [33, 77, 98].

Problem 4.2.2 (Enthalpy weak formulation)**.** *Find* $e \in L^2(D\times(0,T))$ *with* $\beta(e) \in L^2(0,T;H^1(D))$

such that

$$\int_0^T \int_D e\,\phi_t + \nabla_D \beta(e) \cdot \nabla_D \phi + \int_D e_0\, \phi(0) = \int_0^T \int_D f\phi, \tag{4.2.8}$$

for all $\phi \in L^2(0,T;H^1(D)) \cap H^1(0,T;L^2(D))$ *with* $\phi(T) = 0$.

Note that if we assume sufficient regularity of the weak solution $e(x,t)$ and integrate by parts using the calculations (4.2.4) - (4.2.7), we observe a similar cancellation of the free boundary terms, and obtain the following pointwise PDE satisfied by the enthalpy

$$e_t - \Delta_D(\beta(e)) = f. \tag{4.2.9}$$

The existence and uniqueness of a solution to the enthalpy weak formulation, as well as regularity results, have been derived in [55–57] in the case of the flat domain and are required to be adapted to the case of a surface. We will now proceed by outlining the numerical approach presented in [33], based upon an implicit time discretisation and a piecewise linear finite element discretisation of the spatial variables. Here, again we emphasise the results presented below have been adapted to the case of a surface and therefore are required to be verified in the future analysis of the problem.

4.2.2 Finite element approximation

We approximate the smooth surface D by a polyhedral surface

$$D_h = \bigcup_{T \in \mathcal{T}_h} T \qquad \partial D_h = \emptyset,$$

consisting of closed simplices with maximum diameter bounded uniformly above by h, and whose vertices are taken to sit on the smooth surface D. We assume that the triangulation is quasi-uniform, in the sense the in-ball radius $\rho(T)$ of each simplex is uniformly bounded below by $\rho(T) \geq Ch$, for a constant $C > 0$ independent of h, and we furthermore assume that all of the angles in each simplex are less than or equal to $\frac{\pi}{2}$. This acuteness property of the triangulation, was assumed by the authors in [33], to ensure a maximum principle for the finite element approximation of the Laplacian operator. We then consider a piecewise linear finite element space on our triangulation

$$S_h = \{\phi_h \in C^0(D_h) \,|\, \phi_h|_T \text{ is affine linear for all } T \in \mathcal{T}_h\}.$$

Before we introduce an analagous finite element approximation of (4.2.8) to the scheme proposed in [33], we will first introduce some further notation. We denote the discrete L^2-inner product approximating the continuous L^2- inner product on the smooth surface D, by

$$m_h(\phi_h, \chi_h) = \int_{D_h} I_h(\phi_h\, \chi_h) \qquad \text{for } \phi_h, \chi_h \in S_h,$$

where I_h denotes the Lagrangian piecewise linear interpolation operator. We furthermore denote our approximation of the Laplacian term via

$$a_h(\phi_h, \chi_h) = \int_{D_h} \nabla_{D_h}\phi_h \cdot \nabla_{D_h}\chi_h \qquad \text{for } \phi_h, \chi_h \in S_h.$$

We next introduce a time step $\tau = \frac{T}{M}$ for a given $M \in \mathbb{N}_{>0}$, and denote the discrete times via $t_n = n\tau$ for $n = 1, ..., M$. For a given function $g(\cdot, t)$, we set $g^n(\cdot) := g(\cdot, t_n)$ and denote the difference quotient in time via

$$\delta g^n := \frac{g^n - g^{n-1}}{\tau} \qquad n = 1, ..., M.$$

The finite element discretisation proposed in [33], adapted to the case of a deterministic surface then reads as follows.

Problem 4.2.3. *Find $e_h^n, u_h^n \in S_h$ for $n = 0, ..., M$, such that*

$$m_h(\delta e_h^n, \phi_h) + a_h(u_h^n, \phi_h) = m_h(f_h^n, \phi_h) \tag{4.2.10}$$

for all $\phi_h \in S_h$, where $u_h^n = I_h(\beta(e_h^n))$ and $f_h^n = I_h(f(t_n))$. Furthermore, which satisfies the initial condition

$$m_h(e_h^0, \phi_h) = (e_0, \phi_h^l)_{L^2(D)} \quad \textit{for all } \phi_h \in S_h,$$

where ϕ_h^l denotes the lift of ϕ_h, under the closest point projection mapping $a(\cdot)$.

The following error bounds were derived in [33] in the case of a flat domain, and is required to be adapted for the analagous result stated below.

Theorem 4.2.1 (Error bound)**.** *Let $\tilde{e}_h^n$ denote the lift of the discrete approximation e_h^n as defined in (4.2.10). Then we have the following error bounds*

$$\max_n |e(t_n) - \tilde{e}_h^n|_{-1} \leq ch^{\frac{1}{2}}$$

$$\left(\sum_{n=1}^{M} \tau|\beta(e(t_n)) - \beta(\tilde{e}_h^n)|_0^2\right)^{\frac{1}{2}} \leq ch^{\frac{1}{2}}.$$

where $|\eta|_{-1} = (G\eta, \eta)_0^{\frac{1}{2}}$ is the semi-norm with $G : L^2(D) \to H^1(D)$ denoting the Green's operator defined by

$$(\nabla_D G\eta, \nabla_D \phi)_0 = \langle \eta, \phi \rangle \qquad \textit{for all } \phi \in C_0^\infty(D).$$

4.2.3 Numerical implementation

We will now continue by deriving the nonlinear algebraic system of equations equivalent to the finite element approximation (4.2.10), and further discuss an iterative numerical scheme to solve the arising nonlinear system based upon the SOR approach considered in [99]. Let us begin by denoting the nodal basis of the finite element space S_h by $\{\phi_j\}_{j=1,...,N}$. We may then express the discrete approximation e_h^n of the enthalpy at time t_n, and the discrete approximation u_h^n of the temperature as

$$e_h^n = \sum_{j=1}^{N} e_j^n \phi_j \qquad u_h^n = \sum_{j}^{N} u_j^n \phi_j. \tag{4.2.11}$$

We shall denote the respective vectors of coefficients by $e^n = (e_j^n)_{j=1,...,N}$, and $u^n = (u_j^n)_{j=1,...,N}$, and observe that $u_j^n = \beta(e_j^n)$ for $j = 1, ..., N$, as in our finite element approximation we defined $u_h^n = I_h(\beta(e_h^n))$. Substituting in our expressions (4.2.11) for the discrete solutions into our finite element approximation (4.2.10), leads to the following algebraic system, where the entries of the mass $M = (M_{ij})$ and stiffness $A = (A_{ij})$ matrices are prescribed by

$$M_{ij} = m_h(\phi_i, \phi_j) \qquad A_{i,j} = a_h(\phi_i, \phi_j) \qquad \text{for } i, j = 1, ..., N,$$

and where the load vector $F^n = (F_j^n)_{j=1,...,N}$ is given by $F_j^n = m_h(f_h^n, \phi_j)$.

Problem 4.2.4 (Algebraic system). *Given $e_0 \in \mathbb{R}^N$, find $e_n \in \mathbb{R}^N$ for $n = 1, ..., M$, such that*

$$Me^n + \tau A u^n = Me^{n-1} + \tau F^n$$

and where $u^n = (u_j^n)_{j=1,...,N}$ further satisfies $u_j^n = \beta(e_j^n) \quad$ for $j = 1, ..., N$.

It is worth noting the mass matrix M, is diagonal due to the mass lumping considered in our finite element approximation. To numerically compute the solution to the above algebraic system, the authors in [99] propose the following nonlinear over relaxation method. Here we shall denote the iterations approximating the vector of coefficients e^n for the discrete enthalpy e_h^n, by $e_0^n, e_1^n, ...$, and further denote the corresponding components of each iteration by $(e_k^n)_j = e_{j,k}^n$.

Algorithm 2 (A nonlinear SOR method). *The method to compute an approximation of e^{n+1} given e^n.*
Step 1 *Set the initial guess to be $e_0^{n+1} = e^n$ and select a relaxation parameter ω. Note the method was shown to be globally convergent for $\omega \in (0, 2)$.*
Step 2 *Given the previous iteration e_m^{n+1}, successively update its components for $i = 1, ..., N$, via*

$$\begin{aligned} e_{i,m+1}^{n+1} &= (1-\omega)e_{i,m}^{n+1} \\ &\quad + \omega\beta_i^{-1}\left(\tau F_i^{n+1} + M_{ii}e_i^n - \tau\sum_{j<i} A_{ij}\beta\left(e_{i,m+1}^{n+1}\right) - \tau\sum_{j>i} A_{ij}\beta\left(e_{i,m}^{n+1}\right)\right), \end{aligned}$$

where $\beta_i : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function defined by

$$\beta_i(x) = M_{ii}x + \tau A_{ii}\beta(x).$$

Convergence results for the iterative scheme may be found in [99]. Before we continue onto to discuss the next stage, in which we consider the two-phase Stefan on a random surface, we first briefly describe an alternative approach to discretising the nonlinear PDE (4.2.9) satisfied by the enthalpy $e(x, t)$. In contrast to the previous approach taken in [33], which considers an implicit discretisation in time

$$\frac{e^n - e^{n-1}}{\tau} - \Delta_D u^n = 0,$$

and then a further discretisation of the spatial variables via linear finite elements, the authors in [70, 76, 77], instead propose discretising in time using a nonlinear Chernoff formula and combine this with a regularisation of $\beta(\cdot)$

$$u^n - \frac{\tau}{\mu}\Delta_D u^n = \beta_\epsilon(e^{n-1}) \tag{4.2.12}$$

$$e^n = e^{n-1} + \mu(u^n - \beta_\epsilon(e^{n-1})). \tag{4.2.13}$$

Here $0 < \mu \leq 1$ denotes a relaxation parameter, and β_ϵ an approximation of β by a strictly increasing Lipschitz continuous function, with $\epsilon > 0$ denoting the regularisation parameter. The advantage of this an approach is that at each time step, we are only required to solve a linear elliptic PDE and so from a computational perspective, may be quick to implement. It is worth noting that the authors' decision to include the regularisation of $\beta(\cdot)$, in the above approximation arises due to the observed stronger artificial diffusion present in numerical simulations when the regularisation was omitted, see [78] for details. The authors in [77], then combine the discretisation in time, with a finite element discretisation of the spatial variables, adopting piecewise linear elements for the temperature and piecewise constant elements for the enthalpy, leading to a fully discrete linear scheme for which error bounds are subsequently derived, and show a marginal improvement in the convergence rate when compared with the aforementioned scheme in [33].

We now conclude by discussing the next step of generalising the problem to the consideration of a random surface. As with our previous examples, we will let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space, in which Ω is the set of all elementary events for the stochastic domain mapping.

4.2.4 The two-phase Stefan problem over a random surface

Let $D(\omega)$ denote a smooth compact random surface in $\mathbb{R}^3$, which is prescribed by

$$D(\omega) = \{\phi(\omega, x) \,|\, x \in D_{ref}\}$$

for a given vector-valued random field $\phi : \Omega \times \overline{D_{ref}} \to \mathbb{R}^3$, defined over a smooth compact surface $D_{ref} \subset \mathbb{R}^3$. The enthalpy formulation of the two-phase Stefan problem posed over the random surface $D(\omega)$, is then as follows.

Problem 4.2.5 (Stefan problem on a random surface)**.** *Find a random field $e(\omega, x, t)$ defined* *over the random surface $D(\omega)$, i.e. $e(\omega,\cdot,t) : D(\omega) \to \mathbb{R}$, which satisfies for a.e. ω and $t \in (0,T)$*

$$e_t(\omega,x,t) - \Delta_{D(\omega)}\beta(e(\omega,x,t)) = f(\omega,x,t) \quad \textit{in } D(\omega) \tag{4.2.14}$$

$$e(\omega,x,0) = e_0(\omega,x) \quad \textit{in } D(\omega). \tag{4.2.15}$$

Here, the data $f(\omega,\cdot,t)$ and the initial condition $e_0(\omega,\cdot)$ are both given random fields defined over the random surface.

We shall denote the pull-back of a given random field $f(\omega,x,t)$ defined over the random surface $D(\omega)$, i.e. $f(\omega,\cdot,t) : D(\omega) \to \mathbb{R}$, under the domain transformation mapping $\phi(\omega,\cdot)$ by

$$\hat{f}(\omega,x,t) = f(\omega,\phi(\omega,x),t) \qquad x \in D_{ref}.$$

We may now reformulate the two phase Stefan problem (4.2.14) onto the reference surface D_{ref}, using the previously calculated expression for the pull-back of the Laplace-Beltrami operator (2.3.14), to obtain

$$\hat{e}_t(\omega,x,t) - \frac{1}{\sqrt{g_{D_{ref}}(\omega,x)}}\nabla_{D_{ref}} \cdot \left(\sqrt{g_{D_{ref}}(\omega,x)}G^{-1}_{D_{ref}}(\omega,x)\nabla_{D_{ref}}\beta(\hat{e}(\omega,x,t))\right) = \hat{f}(\omega,x,t) \text{ in } D_{ref}$$

$$\hat{e}(\omega,x,0) = \hat{e}_0(\omega,x) \quad \text{in } D_{ref},$$

where the random coefficients are given by

$$\begin{aligned} G_{D_{ref}}(\omega,x) &= \nabla_{D_{ref}}\phi^\top(\omega,x)\nabla_{D_{ref}}\phi(\omega,x) + \nu(x)\otimes\nu(x) \\ g_{D_{ref}}(\omega,x) &= \det(G_{D_{ref}}(\omega,x)), \end{aligned}$$

where $\nu(x)$ denotes the unit normal vector to the reference surface D_{ref}. The weak-formulation over the space-time domain $D_{ref} \times (0,T)$ subsequently reads as follows, where for convenience we will suppress parameters.

Problem 4.2.6 (Enthalpy weak formulation on D_{ref})**.** *For a.e. ω, find $\hat{e}(\omega) \in L^2(D_{ref}\times(0,T))$ with $\beta(\hat{e}(\omega)) \in L^2(0,T;H^1(D_{ref}))$ such that*

$$\begin{aligned} \int_0^T \int_{D_{ref}} \hat{e}(\omega)\,\hat{\phi}_t\sqrt{g_{D_{ref}}(\omega)} + \sqrt{g_{D_{ref}}(\omega)}\,G^{-1}_{D_{ref}}(\omega)\,\nabla_{D_{ref}}\beta(\hat{e}(\omega))\cdot\nabla_{D_{ref}}\hat{\phi} \\ + \int_{D_{ref}} \hat{e}_0(\omega)\,\hat{\phi}(0)\sqrt{g_{D_{ref}}(\omega)} = \int_0^T\int_{D_{ref}} \hat{f}(\omega,t)\hat{\phi}\sqrt{g_{D_{ref}}(\omega)}, \end{aligned} \tag{4.2.16}$$

for all $\hat{\phi} \in L^2(0,T;H^1(D_{ref})) \cap H^1(0,T;L^2(D_{ref}))$ with $\phi(T) = 0$.

We may now choose to discretise the above enthalpy formulation of the Stefan problem reformulated on the reference surface D_{ref}, using a similar approach to [33], whereby we discretise implicitly in time, discretise spatially using piecewise linear elements and further combine this with a Monte-Carlo sampling discretisation of the stochastic variable. This leads to numerically solving the following nonlinear system of algebraic equations over the multiple samples selected and averaging the solutions

$$M(\omega)e^{n+1} + \tau A(\omega)u^{n+1} = \tau F^{n+1}(\omega) + M(\omega)e^{n}.$$

Here the random mass $M(\omega)$ and stiffness $A(\omega)$ matrices and load vector $F^{n+1}(\omega)$ are given by

$$\begin{aligned}
M_{ij}(\omega) &= \int_{D_{ref,h}} I_h\left(\phi_i\,\phi_j\,\sqrt{g^{-l}_{D_{ref}}(\omega)}\right)\\
A_{ij}(\omega) &= \int_{D_{ref,h}} \sqrt{g^{-l}_{D_{ref}}(\omega)}\left(G^{-1}_{D_{ref}}(\omega)\right)^{-l}\nabla_{D_{ref,h}}\phi_i\cdot\nabla_{D_{ref,h}}\phi_j\\
F^{n+1}_j(\omega) &= \int_{D_{ref,h}} I_h\left(\hat{f}(\omega, t^{n+1})\,\phi_j\,\sqrt{g^{-l}_{D_{ref}}(\omega)}\right),
\end{aligned}$$

where f^{-l} denotes the inverse lift onto the discrete approximation $D_{ref,h}$, of the smooth reference surface D_{ref}, under the inverse of the closest point projection mapping $a(\cdot)$. Realisations of the nonlinear system, may each be subsequently solved numerically via the previously mentioned nonlinear SOR method. As a motivating example, numerical computations following this procedure were implemented in DUNE [1, 27], for two realisations of a random surface and have been presented below. The future steps for this particular problem would be a further numerical investigation comparing the efficiency of the two proposed time discretisations previously discussed, namely the implicit time stepping in [33] and the use of the nonlinear Chernoff formula in [77]. This comparison will be necessary, as the sample size required to provide an effective measure of the mean solution increases. Additionally, alternative sampling method, such as the multi-level Monte Carlo method [52, 59], should also be explored to improve computational efficiency of the numerical scheme. After this, the task will be to prove the existence and uniqueness of pathwise solutions to the two phase Stefan problem reformulated on the reference surface, and to establish finite second moments as well as regularity results. The basis for these proofs will be the previous results derived for the deterministic case on a flat domain [55–57]. Following this, an analysis of the selected numerical scheme will need to considered, and additionally an error estimate proved. This will entail a development of the previous arguments presented for the deterministic case [33, 77], with a careful treatment of the stochastic terms, including deriving uniform bounds from the assumptions imposed on the domain mapping, as well as a treatment of the numerical approximation of the surface [42].

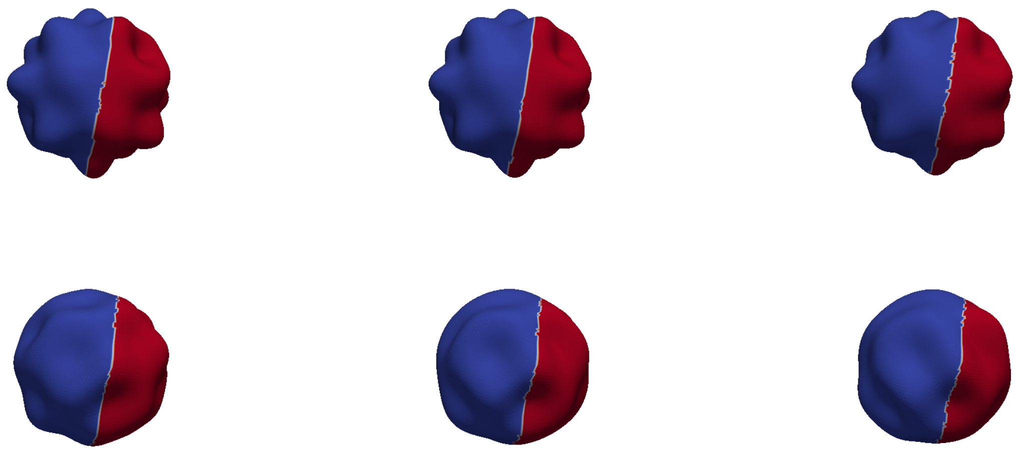

Figure 4.5: The evolution of the free boundary over two realisations of the random surface $D(\omega)$.

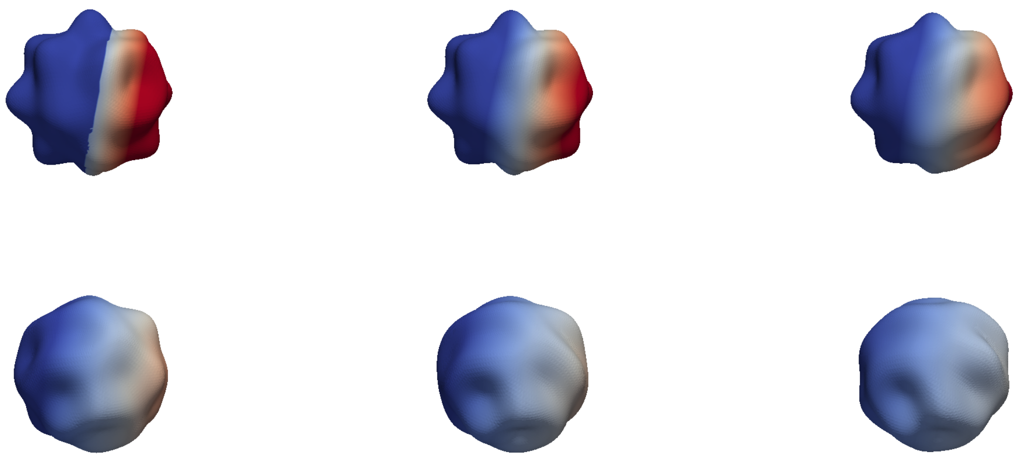

Figure 4.6: The corresponding temperature distributions for the two realisations over time.

Appendix A

Appendix

A.1 Evolving function spaces

In this section, we provide an introduction and overview of the key theorems for evolving Sobolev-Bochner spaces, which will provide a foundation to formulate suitable function spaces to analyse many of the partial differential equations in time-dependent domains under consideration in this thesis. Here, the evolving Bochner spaces will comprise of functions which take values in a time-dependent Hilbert space, as supposed to a fixed Banach space as usually seen with standard Bochner spaces. These functions spaces were first introduced in [95] and subsequently generalised to an abstract framework in [2].

A.1.1 Evolving Lebesgue-Bochner spaces

Let $X = (X(t))_{t\in[0,T]}$ denote a family of real separable Hilbert spaces and $\phi_t : X_0 \to X(t)$ a family of linear homeomorphisms. We denote the initial Hilbert space by $X_0 = X(0)$, the inverse mapping by $\phi_{-t} : X(t) \to X_0$ and introduce the following notion of compatibility of the evolving Hilbert space structure.

Definition A.1.1 (Compatability)**.** *We say that a pair $(X, (\phi_t)_{t\in[0,T]})$ is compatible, provided the following conditions are satisfied:*

1. *(Uniform norm-equivalence) There exists constants $C_1, C_2 > 0$ independent of t such that*

$$C_1\|u_0\|_{X_0} \leq \|\phi_t u_0\|_{X(t)} \leq C_2\|u_0\|_{X_0}, \tag{A.1.1}$$

for all $u_0 \in X_0$ and every $t \in [0,T]$.

2. *(Continuous metric) Furthermore, the mapping*

$$t \mapsto \|\phi_t u_0\|_{X(t)} \tag{A.1.2}$$

is continuous for all $u_0 \in X_0$.

It immediately follows from the above assumptions that the dual operator $\phi_t^* : X^*(t) \to X_0^*$ of the linear homeomorphism $\phi_t : X_0 \to X(t)$, defined by

$$\langle \phi_t^* u, v_0 \rangle_{X_0^*, X_0} = \langle u, \phi_t v_0 \rangle_{X^*(t), X(t)} \qquad \forall v_0 \in X_0,$$

is a linear homeomorphism which satisfies

$$\frac{1}{C_2} \|\phi_t^* u\|_{X_0^*} \leq \|u\|_{X^*(t)} \leq \frac{1}{C_1} \|\phi_t^* u\|_{X_0^*}, \tag{A.1.3}$$

where the inverse is given by ϕ_{-t}^*. Furthermore, due to the assumed separability of X_0, we have that the mapping $t \mapsto \|\phi_{-t}^* f(t)\|_{X^*(t)}$ is measurable for all $f \in X_0^*$. We may now define the evolving Lebesgue-Bochner space as follows.

Definition A.1.2 (Evolving Lebesgue-Bochner spaces)**.** *We define the spaces*

$$L_X^2 = \{u : [0,T] \to \bigcup_{t \in [0,T]} X(t) \times \{t\} \, , \, t \mapsto (\bar{u}(t), t) \,|\, \phi_{-(\cdot)} \overline{u}(\cdot) \in L^2(0,T; X_0)\} \tag{A.1.4}$$

$$L_{X^*}^2 = \{f : [0,T] \to \bigcup_{t \in [0,T]} X^*(t) \times \{t\} \, , \, t \mapsto (\overline{f}(t), t) \,|\, \phi_{(\cdot)}^* \bar{f}(\cdot) \in L^2(0,T; X_0^*)\}, \tag{A.1.5}$$

and equip each function space with the following respective norm

$$\|u\|_{L_X^2} = \left(\int_0^T \|u(t)\|_{X(t)} \, dt \right)^{\frac{1}{2}} \quad \|f\|_{L_{X^*}^2} = \left(\int_0^T \|f(t)\|_{X^*(t)} \, dt \right)^{\frac{1}{2}}.$$

By construction, the push-forward operator $\phi_{(\cdot)}$ defines an isomorphism between $L^2(0,T;X_0)$ and L_X^2, and similarly the dual operator $\phi_{-(\cdot)}^*$ defines an isomorphism between $L^2(0,T;X_0^*)$ and $L_{X^*}^2$. Hence by the norm-equivalences (A.1.1) and (A.1.3), it follows that L_X^2 and $L_{X^*}^2$ are both separable Hilbert spaces. Furthermore, the dual space of L_X^2 may be identified as $\left(L_X^2\right)^* \cong L_{X^*}^2$, where the isometric isomorphism is given through the duality pairing

$$\langle f, u \rangle_{L_{X^*}^2, L_X^2} = \int_0^T \langle f(t), u(t) \rangle_{X^*(t), X(t)}.$$

We next introduce a notion of evolving function spaces which are smooth in time, with the help of the prescribed isomorphism $\phi_t : X_0 \to X(t)$ between the Hilbert space $X(t)$ and the initial Hilbert space.

Definition A.1.3 (Smooth evolving spaces)**.** *For $k \in \mathbb{N}$, we define the spaces*

$$C_X^k = \{\xi \in L_X^2 \,|\, \phi_{-(\cdot)} \xi(\cdot) \in C^k \left([0,T]; X_0\right)\} \tag{A.1.6}$$

$$\mathcal{D}_X(0,T) = \{\xi \in L_X^2 \,|\, \phi_{-(\cdot)} \xi(\cdot) \in C_0^\infty \left((0,T); X_0\right)\} \tag{A.1.7}$$

$$\mathcal{D}_X[0,T] = \{\xi \in L_X^2 \,|\, \phi_{-(\cdot)} \xi(\cdot) \in C^\infty \left([0,T]; X_0\right)\}. \tag{A.1.8}$$

Functions belonging to such spaces possess a natural definition of a material derivative, as the time derivative of the pull-back of the function under the prescribed flow map ϕ_t. For time-dependent domain prescribed by a given parametrisation $\phi(\cdot,t)$, the material derivative takes the form

$$(\partial^\bullet u)(\phi(\cdot,t),t) = \frac{d}{dt}\left(f(\phi(\cdot,t),t)\right) = f_t(\phi(\cdot,t),t) + \nabla f(\phi(\cdot,t),t)\cdot v(\phi(\cdot,t),t)$$

with $v(\phi(\cdot,t),t) = \frac{\partial\phi}{\partial t}(\cdot,t)$. This is generalised to the abstract setting as follows.

Definition A.1.4 (Strong material derivative)**.** *Given $\xi \in C^1_X$, we define the strong material derivative $\partial^\bullet\xi \in C^0_X$ by*

$$(\partial^\bullet\xi)(t) = \phi_t\left(\frac{d}{dt}\left(\phi_{-t}\xi(t)\right)\right).$$

It may easily be verified that if the strong material derivative $\partial^\bullet\xi = 0$ is zero, then there exists some $\eta \in X_0$ such that $\xi = \phi_t\eta$. Conversely, if $\xi = \phi_t\eta$ for a $\eta \in X_0$, then we have $\partial^\bullet\xi = 0$.

In the context of time-dependent partial differential equations, it is common to expect less regularity for the material derivative of the solution compared to the solution itself. We therefore will proceed by introducing a weaker notion of a material derivative for evolving time-dependent Hilbert spaces, based upon integrating by parts

$$\int_0^T \frac{d}{dt}(u,\varphi)_{H(t)}\,dt = 0 \tag{A.1.9}$$

for $\varphi \in \mathcal{D}_V(0,T)$. This will require the identification of the extra term arising from differentiating the inner $(\cdot,\cdot)_{H(t)}$, due to time-dependency of the evolving Hilbert space $H(t)$.

A.1.2 The weak-material derivative and evolving Sobolev-Bochner spaces

Let $V(t) \subset H(t) \subset V^*(t)$ denote a family of Gelfand triples, consisting of separable Hilbert spaces and let $\phi_t : H_0 \to H(t)$ be a family of linear homeomorphisms. We assume that $(H,(\phi_t)_t)$ and the restriction $(V,(\phi_t|_{V_0})_t)$ are both compatible pairs. It therefore follows that

$$L^2_V \subset L^2_H \subset L^2_{V^*}$$

forms a Hilbert triple due to the isomorphism with $L(0,T;V_0) \subset L^2(0,T;H_0) \subset L^2(0,T;V_0^*)$. In order to define the weak-material derivative via the integration by parts (A.1.9), it is necessary to impose the following assumptions on the evolution of the Hilbert structure.

Assumption A.1.1 (Existence of a weak-material derivative)**.** *We assume that for every $u_0, v_0 \in H_0$, the derivative*

$$\hat\lambda(t;u_0,v_0) = \frac{d}{dt}\left(\phi_t u_0,\phi_t v_0\right)_{H(t)} \tag{A.1.10}$$

exists classically. Furthermore, we assume the symmetric bilinear form $\hat\lambda(t;\cdot,\cdot) : H_0 \times H_0 \to \mathbb{R}$ is uniformly bounded

$$|\hat\lambda(t;u_0,v_0)| \le C\|u_0\|_{H_0}\|v_0\|_{H_0} \tag{A.1.11}$$

for a constant $C > 0$ independent of t.

We thus identify the extra term arising from differentiating the inner product $(\cdot,\cdot)_{H(t)}$ solely due to the time-dependency of the Hilbert spaces as follows.

Definition A.1.5 (Time-derivative of the inner product $(\cdot,\cdot)_{H(t)}$). *We define the bounded bilinear form $\lambda(t;\cdot,\cdot) : H(t) \times H(t) \to \mathbb{R}$ by*

$$\lambda(t; u, v) = \hat{\lambda}(t; \phi_{-t}u, \phi_{-t}v). \tag{A.1.12}$$

It may now be easily verified that for any $\sigma_1, \sigma_2 \in C^1_H$, we have

$$\frac{d}{dt}(\sigma_1(t), \sigma_2(t))_{H(t)} = (\partial^\bullet \sigma_1(t), \sigma_2(t))_{H(t)} + (\sigma_1(t), \partial^\bullet \sigma_2(t))_{H(t)} + \lambda(t; \sigma_1(t), \sigma_2(t)). \tag{A.1.13}$$

Consequently, this leads to the following notion of a weak material derivative:

Definition A.1.6 (Weak material derivative). *Given $u \in L^2_V$, we say that u has a weak material derivative provided that there exists a function $\partial^\bullet u \in L^2_{V^*}$ which satisfies*

$$\int_0^T \langle \partial^\bullet u(t), \varphi(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t), \partial^\bullet \varphi(t))_{H(t)} - \int_0^T \lambda(t; u(t), \varphi(t)) \tag{A.1.14}$$

for all $\varphi \in D_V(0,T)$.

Note that if a weak material derivative exists, then it follows from the linearity in (A.1.14) that it must be unique. Addditionally, if a function possess a strong material derivative, then the weak material derivative exists and coincides with the strong material derivative. We may now define the evolving Sobolev-Bochner space.

Definition A.1.7 (Evolving Sobolev-Bochner space). *We define the evolving Sobolev-Bochner space*

$$W(V, V^*) = \{u \in L^2_V \mid \partial^\bullet u \in L^2_{V^*}\}$$

and equip it with the inner product

$$(u, v)_{W(V,V^*)} = \int_0^T (u(t), v(t))_{V(t)} + \int_0^T \langle \partial^\bullet u(t), \partial^\bullet v(t)\rangle_{V^*(t)}$$

Assumption A.1.2 (Sobolev-space equivalence). *We assume that the push-forward operator*

$$\phi_{(\cdot)} : W(V_0, V_0^*) \to W(V, V^*) \tag{A.1.15}$$

defines an isometry with the standard Bochner space

$$W(V_0, V_0^*) = \{u \in L^2(0,T; V_0) \mid u' \in L^2(0,T; V_0^*)\}.$$

Furthermore, we assume that the norms are equivalent, that is there exists constants $C_1, C_2 > 0$

such that

$$C_1\|\phi_{-(\cdot)}u(\cdot)\|_{W(V_0,V_0^*)} \le \|u(\cdot)\|_{W(V,V^*)} \le C_2\|\phi_{-(\cdot)}u(\cdot)\|_{W(V_0,V_0^*)} \tag{A.1.16}$$

for all $u \in W(V,V^*)$.

Under the above assumptions, it follows that $W(V,V^*)$ is a Hilbert space. Furthermore, the following generalisations of the of the standard Bochner-space results hold.

Theorem A.1.1. *The evolving Sobolev-Bochner space $W(V,V_0^*)$ satisfies the following properties:*

- *The embedding $W(V,V^*) \hookrightarrow C_H^0$ is continuous.*
- *The space of smooth functions $\mathcal{D}_V[0,T] \subset W(V,V^*)$ is dense.*
- *Given any $u,v \in W(V,V^*)$, the mapping $t \mapsto (u(t),v(t))_{H(t)}$ is absolutely continuous on $[0,T]$ with*

  $$\frac{d}{dt}(u(t),v(t))_{H(t)} = \langle\partial^\bullet u(t),v(t)\rangle_{V^*(t),V(t)} + \langle\partial^\bullet v(t),u(t)\rangle_{V^*(t),V(t)} + \lambda(t;u(t),v(t))$$

  for a.e. $t \in [0,T]$.

A.1.3 Application: Hilbert spaces on evolving compact surfaces

In this section, we demonstrate how the previous abstract framework may be applied to formulate Sobolev spaces on evolving compact surfaces. In particular, we show how under the standard assumptions imposed on the smoothness of the surface and its evolution, that all the listed assumptions given in the abstract setting are indeed satisfied and thus the evolving function spaces are well-defined. The usual geometric setting considered is as follows.

Let $\{\Gamma(t)\}_{t\in[0,T]}$ denote an evolving compact C^2-hypersurface in $\mathbb{R}^{n+1}$ with initial surface Γ_0. We assume that there exists a map $\phi \in C^1([0,T],C^2(\Gamma_0))$ such that

$$\phi(\cdot,t) : \Gamma_0 \to \Gamma(t)$$

is a diffeomorphic mapping and we define the velocity of $\Gamma(t)$ by

$$v(\phi(\cdot,t),t) = \frac{\partial\phi}{\partial t}(\cdot,t).$$

We further assume that the divergence of the velocity field is uniformly bounded

$$|\nabla_\Gamma \cdot v(x,t)| \le C \quad \forall x \in \Gamma(t), \tag{A.1.17}$$

for a constant $C > 0$ independent of t. We consider the following Hilbert triple

$$H^1(\Gamma(t)) \subset L^2(\Gamma(t)) \subset H^{-1}(\Gamma(t))$$

at each time $t \in [0,T]$, and define the linear homeomorphism $\phi_t : L^2(\Gamma_0) \to L^2(\Gamma(t))$ with the flow map at hand by

$$(\phi_t u_0)(x,t) = u_0(\phi^{-1}(x,t),t) \quad x \in \Gamma(t),$$

for $u_0 \in L^2(\Gamma_0)$ and note that the inverse is given by $\phi_{-t}u = u \circ \phi$ for $u \in L^2(\Gamma(t))$. For convenience, we identify the evolving Hilbert spaces by

$$V = \left(H^1(\Gamma(t))\right)_{t\in[0,T]}, \quad H = \left(L^2(\Gamma(t))\right)_{t\in[0,T]} \quad V^* = \left(H^{-1}(\Gamma(t))\right)_{t\in[0,T]}.$$

Theorem A.1.2. *Under the above assumptions on the smoothness of the surface $\Gamma(t)$ and its evolution, the following results hold:*

- *The pair $(H,(\phi_t)_t)$ and its restriction $(V,(\phi_t|_{V_0})_t)$ are both compatible.*
- *Both of the assumptions (A.1.10) and (A.1.11) to guarantee the existence of a weak-material derivative, are satisfied.*
- *The Sobolev-space equivalence between $W(V_0,V_0^*)$ and $W(V,V^*)$ given in Assumption A.1.2 holds.*

Proof. We first proceed by showing that the pair $(V,(\phi_t|_{V_0})_t)$ is compatible, and note that compatibility of $(H,(\phi_t)_t)$ will follow by a similar argument. Given $u_0 \in H^1(\Gamma_0)$, we have

$$\|\phi_t u_0\|^2_{H^1(\Gamma(t))} = \int_{\Gamma(t)} |\phi_t u_0|^2 + |\nabla_\Gamma(\phi_t u_0)|^2 = \int_{\Gamma_0} \sqrt{g_{\Gamma_0}}|u_0|^2 + \sqrt{g_{\Gamma_0}}G_{\Gamma_0}^{-1}\nabla_{\Gamma_0}u_0 \cdot \nabla_{\Gamma_0}u_0.$$

Here we have substituted in the pull-back of the Laplace-Beltrami operator stated in Lemma 2.3.3. Addditionally, by applying the chain rule to the pull-back $\phi_{-t}u = u \circ \phi$, for a given $u \in H^1(\Gamma(t))$, we have

$$\begin{aligned}\|\phi_{-t}u\|^2_{H^1(\Gamma_0)} \\ &= \int_{\Gamma(t)} \frac{1}{\sqrt{g_{\Gamma_0}\circ\phi^{-1}}}|u|^2 + \frac{1}{\sqrt{g_{\Gamma_0}\circ\phi^{-1}}}\left(\nabla_{\Gamma_0}\phi\circ\phi^{-1}\right)\left(\nabla_{\Gamma_0}\phi^\top\circ\phi^{-1}\right)\nabla_\Gamma u\cdot\nabla_\Gamma u.\end{aligned}$$

We next observe by the regularity of the flow map $\phi \in C^1([0,T],C^2(\Gamma_0))$, that the matrix

$$G_{\Gamma_0} = \nabla_{\Gamma_0}\phi^\top\nabla_{\Gamma_0}\phi + \nu^{\Gamma_0}\otimes\nu^{\Gamma_0},$$

is uniformly continuous since Γ_0 is compact. Therefore, as it is also positive-definite at each (x,t), it follows that there exists constants $C_1, C_2 > 0$ such that

$$C_1|\xi|^2 \le \xi\cdot G_{\Gamma_0}(x,t)\xi \le C_2|\xi|^2 \quad C_1 \le \sqrt{g_{\Gamma_0}(x,t)} \le C_2$$

for all $x \in \Gamma(t), t \in [0,T]$ and every $\xi \in \mathbb{R}^{n+1}$. Hence the uniform norm-equivalence (A.1.1) is satisfied. Furthermore, the mapping $t \mapsto \|\phi_t u_0\|_{H^1(\Gamma(t))}$ for $u_0 \in H^1(\Gamma_0)$ is continuous due to uniform continuity of $\sqrt{g_{\Gamma_0}}$ and $G_{\Gamma_0}^{-1}$. Thus $(V,(\phi_t|_{V_0})_t)$ is compatible. For the existence of a weak material derivative, we argue by the transport formulae, that the following derivative exists classically

$$\hat{\lambda}(t; u_0, w_0) = \frac{d}{dt}\int_{\Gamma(t)} \phi_t u_0 \, \phi_t w_0 = \int_{\Gamma(t)} \phi_t u_0 \, \phi_t w_0 \, (\nabla_\Gamma \cdot v)$$

which confirms (A.1.10). Furthermore, the uniform bound (A.1.17) on the divergence of the velocity of $\Gamma(t)$, implies that the bilinear form $\hat{\lambda}(t; \cdot, \cdot)$ is uniformly bounded as required in (A.1.11). It remains to prove the equivalence between the Sobolev spaces $W(V, V^*)$ and $W(V_0, V_0^*)$ stated in Assumption A.1.2.

Given any $u \in W(V_0, V_0^*)$, we have by construction of the evolving Sobolev-Bochner space $W(V, V^*)$ that $\phi_{(\cdot)}u(\cdot) \in L^2_V$. Hence we are only required to prove that $\phi_{(\cdot)}u(\cdot)$ possess a weak-material derivative which belongs to the space $L^2_{V^*}$. Considering the pull-back of the right hand side in (A.1.14) for an arbitrary $\varphi \in W(V, V^*)$, yields

$$\begin{aligned}
&\int_0^T (\phi_t u(t), \, \partial^\bullet \varphi(t))_{H(t)} + \int_0^T \lambda(t; u(t), \varphi(t)) \\
&\quad = \int_0^T (u(t), \sqrt{g_{\Gamma_0}(t)}\, (\phi_{-t}\varphi(t))'\,)_{H_0} + \int_0^T (u(t), \left(\sqrt{g_{\Gamma_0}(t)}\right)' \, \phi_{-t}\varphi(t)\,)_{H_0}.
\end{aligned}$$

Here we have used the identity $\sqrt{g_{\Gamma_0}}\,(\nabla_\Gamma \cdot v) \circ \phi = \partial_t \sqrt{g_{\Gamma_0}}$, as well as the property that when we pull-back the inner product $(\cdot, \cdot)_{L^2(\Gamma(t))}$ onto the reference surface Γ_0 introduces the surface area element $\sqrt{g_{\Gamma_0}}$. Collecting the terms, observing $\sqrt{g_{\Gamma_0}} \in W^{1,\infty}$ by the regularity of the flow mapping and recalling that u is weakly differentiable, we deduce

$$= \int_0^T (u(t), (\sqrt{g_{\Gamma_0}(t)}\, \phi_{-t}\varphi(t))'\,)_{H_0} = -\int_0^T \langle u'(t), \sqrt{g_{\Gamma_0}(t)}\, \phi_{-t}\varphi(t) \rangle_{V_0^*, V_0}.$$

Hence it follows that $\phi_{(\cdot)}u(\cdot)$ possess a weak material derivative given by

$$\partial^\bullet \left(\phi_{(\cdot)}u(\cdot)\right) = \phi^*_{-t}\left(\sqrt{g_{\Gamma_0}(t)}u'(t)\right)$$

which belongs to $L^2_{V^*}$ since $\sqrt{g_{\Gamma_0}(t)} \in W^{1,\infty}$. By a similar arguement, one may show that if given $u \in W(V, V*)$, then the pull-back $\phi_{-(\cdot)}u(\cdot)$ is also weakly differentiable, with derivative equal to $\frac{1}{\sqrt{g_{\Gamma_0}(t)}}\phi^*_t\,(\partial^\bullet u(t))$. We therefore have established the required equivalence between Sobolev-Bochner spaces. $\square$

A.2 Fundamental calculus results on surfaces

In this section, we provide proofs of some the key theorem and formulae relating to fundamental calculus results on surfaces, that are frequentely referred to throughout this thesis. Note that the proofs of the results below originate from the work found in [42]. We begin by deriving a formula for integration by parts on a surface.

Theorem A.2.1 (Integration by parts)**.** *Given a sufficiently smooth hypersurface* $\Gamma \subset \mathbb{R}^{n+1}$ *with boundary* $\partial\Gamma$*. Then for any* $f \in C^1(\bar{\Gamma})$ *we have*

$$\int_\Gamma \nabla_\Gamma f \, dA = \int_\Gamma f H \nu dA + \int_{\partial\Gamma} f \mu dA \tag{A.2.1}$$

where μ *denotes the co-normal vector, which is normal to the boundary and tangent to the surface.*

Proof. Let us extend f to an open neighbourhood around Γ as follows

$$\bar{f}(x) = f(a(x)) \quad x \in U_\epsilon$$

where the open set is defined as $U_\epsilon = \{x \in \mathbb{R}^{n+1} \,|\, |d^\Gamma(x)| < \epsilon\}$ for $\epsilon > 0$. By Gauss's theorem, we have

$$\int_{U_\epsilon} \bar{f}(x) \, dx = \int_{\partial U_\epsilon} \bar{f}(x) \nu_{\partial U_\epsilon}(x) \, dx.$$

We may now express the integrals in terms of integrals over hypersurfaces $\Gamma(\epsilon) = \{x \in \mathbb{R}^{n+1} \,|\, d^\Gamma(x) = \epsilon\}$ parallel to Γ with the help of the co-area formula

$$\frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Gamma(\epsilon)} \nabla \bar{f}(x) = \frac{1}{2\epsilon} \left(\int_{\Gamma(\epsilon)} \bar{f}(x) \nu(x) - \int_{\Gamma(-\epsilon)} \bar{f}(x) \nu(x) \right) + \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\partial\Gamma(\delta)} \bar{f}(x) \mu(x).$$

Computing the derivative of the extension $\bar{f}(x) = f(a(x))$ by the chain rule gives

$$\nabla \bar{f}(x) = \left(I + d^\Gamma(x) \mathcal{H}^\Gamma(a(x)) \right)^{-1} \nabla_\Gamma f(a(x)).$$

Consequently, we deduce the limit of the first integral is as follows

$$\frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Gamma(\delta)} \nabla \bar{f}(x) = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\Gamma(\delta)} \left(I + \delta \mathcal{H}^\Gamma(a(x)) \right)^{-1} \nabla_\Gamma f(a(x)) \to \int_\Gamma \nabla_\Gamma f.$$

Similarly, the limit of the last integral becomes

$$\frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} \int_{\partial\Gamma(\delta)} \bar{f} \mu \to \int_{\partial\Gamma} f \mu.$$

For the remaining integrals, we observe with the transport property that

$$\frac{d}{d\epsilon} \int_{\Gamma(\epsilon)} f \nu \big|_{\epsilon=0} = \int_\Gamma f \nu \nabla \cdot \nu = \int_\Gamma f H \nu$$

since we may consider $\Gamma(\epsilon)$ as an evolving surface with velocity given by $v = \nu$. $\square$

We continue with the Leibniz integration rule for the time derivative of integrals over time-dependent domains, applied to the case of evolving surface integrals.

Theorem A.2.2 (Leibniz integration rule for evolving surfaces)**.** *Let* $\Gamma(t)$ *denote a smooth evolving hypersurface in* $\mathbb{R}^{n+1}$ *with boundary* $\partial\Gamma(t)$*, which evolves under a given velocity field* $v = v_\nu + v_\tau$*. Then we have*

$$\frac{d}{dt}\int_{\Gamma(t)} f\,dA = \int_{\Gamma(t)} \partial^\bullet f + f\nabla_\Gamma \cdot v\,dA. \tag{A.2.2}$$

Proof. Let $X : U \to V \cap \Gamma(t)$ denote a local parametrisation of a portion of $\Gamma(t)$ evolving by the prescribed velocity field v, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n+1}$ are open sets. Furthermore, let us define

$$F(\theta, t) = f(X(\theta, t), t) \quad \theta \in U$$

and denote the first fundamental form by $G = \nabla X^\top \nabla X$ and its determinant by $g = det(G)$. The time derivative of the surface integral may be expressed in local coordinates as follows

$$\frac{d}{dt}\int_{\Gamma(t)\cap V} f = \frac{d}{dt}\int_U F\sqrt{g} = \int_U F_t\sqrt{g} + F\left(\sqrt{g}\right)_t.$$

By Jacobi's formula for the derivative of a determinant and the symmetry of the first fundamental form we deduce

$$\left(\sqrt{g}\right)_t = \frac{1}{2\sqrt{g}} g \operatorname{trace}\left(G^{-1}\frac{\partial G}{\partial t}\right) = \sqrt{g}\operatorname{trace}\left(G^{-1}\nabla X_t \nabla X\right).$$

It may be subsequently verified with the local coordinate expression for the tangential derivative of a function $\nabla_\Gamma f \circ X = \nabla X G^{-1}\nabla F$, and the fact $v \circ X = X_t$, since by assumption the local parametrisation evolves with the given velocity field, that we have

$$\operatorname{trace}\left(G^{-1}\nabla X_t \nabla X\right) = (\nabla_\Gamma \cdot v) \circ X$$

which consequently leads to the stated result. □

It is worth noting, that in case where $\Gamma(t)$ is an evolving compact hypersurface and thus without a boundary, that the above integral on the right hand side of (A.2.2) is independent of the tangential velocity field of $\Gamma(t)$. If we express the given velocity field into its normal and tangential component $v = v_\nu + v_\tau$, then this property may be observed by first noting that

$$\int_{\Gamma(t)} \partial^\bullet f + f\nabla \cdot v = \int_{\Gamma(t)} \partial^\circ f + f\nabla_\Gamma \cdot v_\nu + f\nabla_\Gamma \cdot v_\tau + \nabla_\Gamma f \cdot v_\tau$$

and then integrating by parts to find that the last two terms vanish

$$\int_{\Gamma(t)} \nabla_\Gamma \cdot (f v_\tau) = \int_{\Gamma(t)} f(v_\tau \cdot \nu) = 0.$$

A.3 Tensor structure of separable Hilbert spaces

In this section, we discuss some of the key results relating to the tensor structure of separable Hilbert spaces, that was exploited in our analysis of advection-diffusion equations on randomly evolving domains. However, we first begin by considering the separability of the space of real-valued random variables $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ for a general complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which for convenience shall be denoted by $L^p(\Omega)$. Although the measure is $\sigma-$finite, this does not necessarily gaurantee that the space $L^p(\Omega)$ is separable. In fact, counter-examples may quickly be constructed based upon a overly large $\sigma-$algebra of events. For instance, if take $\Omega = \mathbb{R}, \mathcal{F} = \mathcal{B}(\mathbb{R})$ and $\mathbb{P}(E) = |E|$ to be the counting measure, then it may easily be verified that there does not exists a countable dense subset of $L^2(\Omega)$. We therefore restrict the relative size of the $\sigma-$algebra $\mathcal{F}$, by introducing the following notion of separability for a probability space.

Definition A.3.1 (Separability)**.** *We say that a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is separable provided that* $\mathcal{F}$ *is generated by a countable collection of subsets.*

This leads to the desired separability of the space $L^p(\Omega)$, for which a proof of this theorem may be found in [8, Theorem 4.13].

Theorem A.3.1 ($L^p(\Omega)$- separability)**.** *Given a complete separable probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, we have that* $L^p(\Omega)$ *is separable for* $1 < p < \infty$*.*

We next continue, by considering the tensor structure of separable Hilbert spaces.

A proof may be found in [87].

Theorem A.3.2 (Tensor structure)**.** *Let* (X, μ) *and* (Y, ν) *denote measure spaces such that* $L^2(X, \mu)$ *and* $L^2(Y, \nu)$ *are both separable, and let* H *denote a real separable Hilbert space. Then we have the following isometric isomorphisms*

$$L^2(X, \mu) \otimes L^2(Y, \nu) \cong L^2(X \times Y, \mu \times \nu) \tag{A.3.1}$$

$$L^2(X, \mu) \otimes H \cong L^2(X, \mu; H). \tag{A.3.2}$$

A.4 Assumptions reference page

We now provide as a reference to the reader, the page numbers of the assumptions imposed on our proposed finite element discretisations of the elliptic problem in Chapter 2 and the advection-diffusion problem in Chapter 3.

A.4.1 Assumptions on the finite element discretisation of the elliptic problem

- (L1-L2) - pg. 26
- (I1) - pg. 27

- (P1-P3) - pg. 27
- (R1) - pg. 27.

A.4.2 Assumptions on the finite element discretisation of the advection-diffusion problem

- $(M_h1 - M_h2)$ - pg. 85
- $(A_h1 - A_h3)$ - pg. 85
- $(B_h1 - B_h2)$ - pg. 85
- $(T_h1 - 3)$ - pg. 85
- $(L1 - L2)$ - pg. 88
- $(T_h^l1 - 3)$ - pg. 90
- (M^l2) - pg. 90
- (A^l3) - pg. 90
- (B^l2) - pg. 90
- (I) - pg. 90
- (G1 - 9) - pg. 91-92
- (V1 - 4) - pg. 92
- (R) - pg. 92.

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